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A SHORT
TREATISE
ON THE
CONIC SECTIONS;

IN WHICH
THE THREE CURVES ARE DERIVED FROM A
GENERAL DESCRIPTION ON A PLANE,
AND
THE MOST USEFUL PROPERTIES OF EACH ARE
DEDUCED FROM A COMMON PRINCIPLE.

BY THE REV. T. NEWTON, M.A. K

FELLOW OF JESUS COLLEGE, CAMBRIDGE

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IN SENATE

BY THE SENATE

APPROVED

THIS 14th DAY OF

SEPTEMBER

1881

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JOHN A. BROWN

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P R E F A C E.

THE following Treatise was drawn up for the use of the author's pupils; and the only motive which induced him to publish it, was a desire of promoting the study of Conic Sections in this University, by facilitating the attainment of that useful branch of geometry, which has not been sufficiently attended to. Some enter upon Newton's Principia with little or no previous knowledge of conics, taking the common properties of the sections for granted. Others, who have once laboured through some kind of demonstration, think it sufficient, if they remember the propositions, although they have forgotten the proofs. Such inattention to a science, which is of so much importance in the present system of philosophy, I can attribute only to the want of a concise geometrical treatise.

The authors who have written upon the sub-

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ject may be divided into two classes; those who have begun with the sections of the cone; and those who have begun with a description of the curves in plano. Although some of the demonstrations of the latter, who have treated the subject geometrically, are short and perspicuous, yet there are others, upon which depend some of the principal properties, that are tedious and difficult. The demonstrations of the first class of authors are free from this objection, being in general plain and concise; but they have been obliged to introduce so many previous propositions, concerning the properties of lines touching and cutting conical surfaces, in order to arrive at the principal properties of the three sections, that it requires more time, than can well be spared from that portion which is allotted to an academical education, and more resolution than most young men are possessed of, to go through them. Those of the second class, who have treated the subject algebraically, have, some of them, reduced the whole into a narrower compass; but, in their eagerness to avoid the charge of prolixity, they have fallen into another more exceptionable fault. The method in which they have deduced some of the properties, particularly the relations of the ordinates and abscissæ, is extremely operose and inelegant; each
step

step in the proofs is so little connected with the preceding one, that it is scarcely possible to retain them in the memory.

I shall not detain the reader with any comparison between the two methods, in order to determine which is the best; much may be said in favour of both; but the preference seems to have been given to the latter, at least in this University, where mathematical and philosophical studies are more particularly attended to; yet, with regard to neatness and clearness of demonstration, the authors of the latter class cannot be compared with some of the former. To be convinced of this, let the reader take any one of the principal properties of the ellipse or hyperbola, and compare the algebraical demonstration of L'Hospital, or Trevigar, the geometrico-algebraical demonstration of Emerson, or even the geometrical one of Simson with that of Hamilton, and he will not hesitate to determine in favour of the latter. But the sections of the cone, on account of the many intersections of right lines and planes with surfaces, and with solids, are not easily to be comprehended by those who are acquainted only with the elements of geometry.

I had long been persuaded, that all the known properties of the conic sections
a 2 might

might be deduced geometrically from a principle which is common to all the sections, viz. the given ratio between the distances of every point in the curve from the focus and the directrix, in the same manner as Hamilton has deduced some of them in the second book of his treatise. I therefore assumed that property, which he has demonstrated, Prop. 11. Book 2. for a definition of a conic section, and had made a considerable progress, before I was fortunate enough to meet with the *Elementa Matheseos* of Boscovich; a work which seems to have been little known, or not so much esteemed as it deserves, although the author is justly celebrated for his later productions. In his elements of conic sections, which have all the advantages of those authors who have begun with the cone, without any of the disadvantages, I found the plan I had in view in a great measure executed. I have, therefore, adopted many of his demonstrations, with little or no variation; the arrangement of the propositions, and several of the proofs have been much altered; and of some I have been obliged to give new demonstrations, having excluded the harmonical division of right lines, upon which they depended. I have also taken from other authors, particularly from the second book of Hamilton, such

P R E F A C E.

such propositions as were conformable to the present plan. Upon the whole, I have endeavoured to compress the subject as much as possible, and yet not to omit any of the properties, which every one ought to be acquainted with, previous to his entering upon the Principia of Newton, and the branches of natural philosophy; I have also taken care that the demonstrations should be strictly geometrical, such as the young student will find no difficulty in understanding, provided he be well acquainted with the Elements of Euclid, and plane Trigonometry.

As this treatise was designed to be an introduction to the Principia, I could not, with propriety, make use of the principles contained in the first section, in comparing the areas of the sections which have a common axis, or in the quadrature of the parabola; but if the reader be already acquainted with the doctrine of ultimate ratios, he may shorten the demonstration of proposition 69. in the following manner. It being proved, that the parallelograms in the figure APN are to the parallelograms in the figure AQN in the given ratio of NP to NQ , which is the same with that of the conjugate axes, and the parallelograms APN , AQN being ultimately equal to the areas APN , AQN , Lem. 2.

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New-

Newton's Principia, the areas are to each other in the same ratio.

The quadrature of the parabola may be demonstrated from the same principles. Let AN , Pl. 10. Fig. 88. be the abscissa of any diameter of a parabola, and PNQ the ordinate; through the point A draw BC parallel to PQ ; and through P, Q draw PB, QC parallel to NA ; then the area PAQ will be to the parallelogram $PBCQ$ as 2 to 3; for let the abscissa AN be divided into any number of equal parts, of which ND is one; through D draw HI parallel to PQ , cutting the parabola in the points F, G ; and through the point F draw KE parallel to NA ; take KR equal to KP , and draw RL parallel to NA ; then the parallelogram RB will be double the parallelogram KB ; and if the number of parts in AN be increased without limit, the parallelogram DP will be equal to the parallelogram RB : for the right line HG will be ultimately equal to PQ , or $2PN$; and, Prop. 31. the rectangle under HF, HG , or the rectangle under PK and $2PN$, is to PH as the square of PN is to NA , or PB ; and alternately, the rectangle under PK and $2PN$, or the rectangle under $2PK$ and PN , is to the square of PN as PH is to PB ; therefore $2PK$, or PR , is to PN as PH is to PB , and the parallelograms RB, PD are equiangular; there-

therefore they are equal, and the parallelogram PD is to the parallelogram KB as 2 to 1, and the sum of all the parallelograms in APN to the sum of all in APB , or the area APN to the area APB in the same ratio. Therefore the area APN is to the parallelogram $ABPN$ as 2 to 3, and the area PAQ to the parallelogram $PBCQ$ in the same ratio.

I have only to add that, had it not been for the encouragement, and the liberal assistance which I have received from the University, in defraying the expence of paper and printing, this little tract would not have appeared in public.



Therefore the present state of the world is
P.D. 5 to the present time. It is a state
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CONIC SECTIONS.

DEFINITIONS.

I. IF any point S be assumed without the line DX , and whilst the line SP revolves about S as a Center, a point P moves in it in such a manner, that its distance from the point S shall always be to PE , its distance from the line DX , in a given ratio, the Curve described by the point P is called a Conic Section; a Parabola, an Ellipse, or an Hyperbola, according as SP is equal to, less, or greater than PE . FIG. 1,
2.

II. The indefinite right line DX is called the Directrix.

III. The point S is called the Focus.

IV. The ratio of SP to PE is called the Determining Ratio.

V. If a line SD be drawn through the Focus perpendicular to the Directrix, which is produced indefinitely, it is called the Axis of the Conic Section.

VI. The point A , where the Curve meets the Axis, is called the Vertex.

VII. A right line LST , drawn through the Focus parallel to the Directrix, and terminated by the Curve in the points L , T , is called the Principal Parameter, or the Latus Rectum.

A

COR.

FIG. 2.

COR. 1. SP being greater than PE in the hyperbola, two curves will be described, one on each side of the directrix; which are called opposite hyperbolas.

COR. 2. When the revolving line SP comes into the position SAD , SP , PE will be equal to SA , AD ; therefore SA is to AD in the determining ratio.

COR. 3. When the line SP comes into the position SL , or ST , the distance of P from the directrix will be equal to SD , and SL , or ST will be to SD in the determining ratio; and therefore the latus rectum LT is bisected in S .

COR. 4. The latus rectum in the parabola is equal to twice the distance of the focus from the directrix, or to four times its distance from the vertex. For SL is equal to SD , and SA is equal to AD ; therefore LT is equal to twice SD , or to four times SA .

PROPOSITION I.

FIG. 3.
4.
5.

If two right lines DLQ , DTq be drawn from the point D , where the axis meets the Directrix, through L and T , the extremities of the Latus Rectum, which are produced both ways in the Hyperbola; and through any point P in the Conic Section a line QPq be drawn parallel to the Directrix, meeting the two lines DLQ , DTq in Q and q ; the Segment QN , which is intercepted between either of the lines and the axis, will be equal to SP , the distance of P from the Focus.

FOR

FOR the triangles DNQ , DSL are similar, and NQ is to ND as SL is to SD , that is, Cor. 3. Def. in the determining ratio, or as SP to ND ; therefore NQ is equal to SP . In the same manner it may be proved that qN is equal to Sp , which is equal to SP , each of them being to ND in the determining ratio.

COR. 1. If KAG be drawn through the vertex parallel to the directrix, SA will be equal to AK , or AG .

COR. 2. The lines DLQ , DTq touch the Conic Section in the points L and T . For SNP being a right angled triangle, SP is always greater than PN , except when P is at L , where they coincide; therefore QN is always greater than PN , except when QN coincides with LS ; DQ , therefore, meets the curve only in one point L . In the same manner it may be shown that Dq touches the curve in T .

COR. 3. The angle LDT , contained between the tangents DLQ , DTq , is a right angle in the parabola, an acute angle in the ellipse, and an obtuse angle in the hyperbola. For in the parabola SL is equal to SD , and the angle DSL is a right angle; therefore the angle SDL is half a right angle, and the whole angle LDT a right angle. In the ellipse SL is less than SD , and the angle SDL less than the angle SLD , therefore less than half a right angle, and the whole angle LDT less than a right angle. In the hyperbola SL is greater than SD , and the angle SDL greater than SLD , therefore greater than half a right angle, and the whole angle LDT greater than a right angle.

FIG. 3.

FIG. 4.

FIG. 5.

P R O P. II.

FIG. 3. If from the point G , where the right line
 4. KAG , which is drawn through the Vertex
 5. parallel to the Directrix, meets either of the
 Tangents DTq , DLQ , a line GSR be
 drawn through the Focus, which is produced
 both ways in the Hyperbola, it will be parallel
 to the other Tangent DLQ in the Parabola;
 it will meet it somewhere in g , in the direc-
 tion Gsg , in the Ellipse, and in the oppo-
 site direction in the Hyperbola.

LET SG meet the directrix in X ; and because
 the triangles SAG , SDX are similar, and SA ,
 Cor. 1. Prop. 1. is equal to AG , SD will be equal
 FIG. 3. to DX ; but, in the parabola, SL is equal to SD ;
 it is, therefore, equal and parallel to DX ; and con-
 FIG. 4. sequently, XGS is equal and parallel to DL . In
 the ellipse SL is less than SD , or DX ; and there-
 FIG. 5. fore the lines DL , XS must meet, when produced,
 in the direction XGS . In the hyperbola SL is
 greater than SD , or DX ; and therefore the lines
 must meet, when produced, in the direction SGX .

COR. 1. Because the triangle SNR is similar to
 the triangle SAG , SN will be equal to NR .

FIG. 4. COR. 2. Hence when Q coincides with g , in the
 5. ellipse or opposite hyperbola, QN will be equal
 to gM , or SM ; therefore SP will be equal to SN ;
 and therefore SP will coincide with SN , and the
 curve will meet the axis in the point M .

FIG. 4. COR. 3. Hence the whole ellipse is contained
 o between

between the lines GK , gk , on one side of the directrix.

COR. 4. In the parabola, NL being always greater than NR , except at the vertex, SP is greater than SN ; therefore the curve will meet the axis only in one point A , and it will be extended without limit, on one side of the directrix. FIG. 3.

COR. 5. In the Hyperbola, NL being greater than NR , except at A and M , SP is greater than SN , and the two curves will be extended without limit, on opposite sides of the directrix. FIG. 5.

COR. 6. The lines KAG , kMg touch the Conic Section in the points A and M . For P and p coincide in these points; therefore the right lines KAG , kMg meet the curve only in one point.

COR. 7. In the ellipse, SP is the greatest distance from the focus, and SA the least; and those distances which are nearer to SM are greater than those which are more remote. For SP is to ND in a given ratio; ND increases from A to M , it is the least at A , and the greatest at M ; therefore SP increases from A to M in the same proportion. FIG. 4.

COR. 8. In the parabola and hyperbola SA is the least distance from the focus, and SP increases without limit; and SM is the least distance in the opposite hyperbola.

DEFINITIONS.

VIII. The tangents DLQ , DTq , which are drawn through the extremities of the Latus Rectum, are called Focal Tangents.

IX. The right line AM , in the Ellipse and Hyperbola, is called the Transverse Axis, or the Axis Major. FIG. 4.
5.

X. If the Transverse Axis be bisected in C , the point

point C is called the Center of the Ellipse or Hyperbola.

XI. If a line BCb , which is bisected in C , be drawn perpendicular to the Transverse Axis, and CB , Cb be each of them a mean proportional between SA , SM , the segments of the axis intercepted between the focus and the vertices, BCb is called the Conjugate Axis, or the Axis Minor.

XII. A right line PNp , drawn through any point N in the Axis parallel to the Tangent KAG , or perpendicular to the Axis, and terminated by the Curve in the points P and p , is called an Ordinate to the Axis.

XIII. And the Segment of the axis AN , intercepted between the ordinate and the vertex, in all the Sections, as also the other segment NM in the Ellipse and Hyperbola, is called an Abscissa.

XIV. Any line passing through the center of an Ellipse or Hyperbola, which is terminated both ways by the Curve in the former, and by the opposite Curves in the latter, is called a Diameter.

XV. A line drawn through any point in the parabola parallel to the Axis is called a Diameter to the Parabola.

XVI. Any point where a Diameter meets the Curve is called a Vertex to that Diameter.

P R O P. III.

The Axis bisects all its Ordinates, and divides the Conic Section into two equal and similar parts.

FIG. 3. **L**ET PNp be any ordinate, meeting the axis
 4. in N . Join SP , Sp ; and because Sp is equal
 5. to

to SP , Prop. 1. and SN is common to the two right angled triangles SNP , SNp , Np will be equal to NP . And because all the ordinates are bisected, if the curve ATp be turned round upon the axis AN , and placed upon ALP , all the points p will coincide with all the points P , and the curve ATp with the curve ALP .

PROP. IV. PROB.

The Focus, Directrix, and the Determining Ratio being given, to describe the Conic Section.

LET DX be the directrix, and S the focus. FIG. 3.
 Through the point S draw SD perpendicular 4.
 to DX , which produce indefinitely. Draw LST 5.
 parallel to DX ; and take SL and ST to SD in the
 determining ratio. Then LST , Cor. 3. Def. is
 the latus rectum. Join DL , DT , which must
 also be produced indefinitely. Take DX , in the
 directrix, equal to DS , and join XS , cutting the
 line DT in G , which, Prop. 2. will be parallel to
 the line DL in the parabola; it will meet it in some
 point g , in the direction DL , in the ellipse, and in
 the opposite direction in the hyperbola. Through
 the points G and g draw KAG , gMk parallel to the
 directrix, meeting the two lines DLg , DTk , and
 the axis in the points K , G , A , and g , k , M ; the
 points A and M will, therefore, be the vertices of
 the axis. Through any point N , in the axis, be-
 tween A and M in the ellipse, any where on the
 same side of A with S in the parabola, and any
 where except between A and M in the hyperbola,
 draw the line QNg parallel to the directrix; and
 from

from the center S , with a radius equal to QN , describe a circle, cutting the line Qq in the points P, p ; and join SP, Sp , which are each of them equal to QN ; and therefore the points P and p will be in the curve, by the first proposition.

COR. 1. If the latus rectum, and the distance of the focus from the vertex be given, the Conic Section may be described. For let S be the focus; and through S draw the indefinite right line ASN , in which take SA equal to the given distance from the vertex. Draw LST perpendicular to ASN ; and take SL, ST each of them equal to half the latus rectum. Through A draw KAG parallel to LT ; and take AG equal to AS . Join TG , which produce to meet SA in D : Join DL and GS , which produced, will meet in the point g in the direction DL , in the opposite direction, or they will be parallel; in the first case the Section will be an ellipse, in the second an hyperbola, and in the third a parabola, and the curve may be described as before.

COR. 2. Hence if two right lines DQ, Dq be inclined to each other at any given angle, which is bisected by the line DN , and another line Qq , which is perpendicular to DN , move parallel to itself, and intersect the lines SP, Sp , revolving about any point S in the line DN , in such a manner, that SP, Sp shall always be equal to QN or qN , the points of intersection P, p will describe a Conic Section, which, Cor. 3. Prop. 1. will be a parabola, an ellipse, or an hyperbola, according as the angle QDq is equal to, less, or greater than a right angle.

PROP.

P R O P. V.

The square of the semi-ordinate to the axis;
in the Parabola, is equal to the rectangle
under the latus rectum and the abscissa.

BECAUSE the line Qq is bisected in N , the rectangle QRq , together with the square of RN , is equal to the square of QN , which is equal to the square of SP , Prop. 1. or to the squares of SN , PN , or of RN , PN , Cor. 1. Prop. 2. and if from each of these be taken the square of RN , the remaining rectangle QRq will be equal to the square of PN . But QR is equal to SL , and Rq is equal to RN and Nq , or to SN , Nq ; and the angle NDq being half a right angle, Cor 3. Prop. 1. NqD is also half a right angle, and Nq is equal to ND ; therefore Rq is equal to SN and ND , that is to SD and twice SN , or to twice SA and twice SN , Cor. 2. Def. which is equal to twice AN ; therefore the rectangle QRq is equal to the rectangle under SL and twice AN , or to the rectangle under AN and twice SL , or LT ; and therefore the square of PN is equal to the rectangle under AN , LT .

FIG. 3.

COR. 1. The latus rectum being constant, the abscissa will vary as the square of the ordinate.

COR. 2. The parabola recedes from the axis without limit. For the abscissa increases without limit, and therefore the square of the semi-ordinate, which varies as the abscissa, will also increase without limit.

COR. 3. Any line, which is drawn parallel to the axis of the parabola, will cut the curve only in one point.

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point. For if it be supposed to cut the curve in more points than ~~one~~, the semi-ordinates drawn through the points of intersection would be equal, when the abscissæ are unequal, which is impossible.

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If four straight lines be proportionals, and any other four proportionals, the rectangle under the first and fifth will be to the rectangle under the second and sixth as the rectangle under the third and seventh to the rectangle under the fourth and eighth.

FIG. 6.

LET AB be to CD as EF to GH , and BI to DK as FL to HM ; and let AI be the rectangle under AB , BI ; CK the rectangle under CD , DK ; EL the rectangle under EF , FL ; and GM the rectangle under GH , HM ; then AI will be to CK as EL to GM . For in DK , HM , produced if necessary, take DN , HO such, that AB shall be to CD as DN to BI , and EF to GH as HO to FL ; and complete the rectangles CN , GO . Then the rectangle CN is equal to the rectangle AI , and the rectangle GO equal to the rectangle EL . But, AB being to CD as EF to GH , and as DN to BI , DN is to BI as EF to GH , or as HO to FL . But BI is to DK as FL to HM ; therefore DN is to DK as HO to HM ; and the rectangle CN being to the rectangle CK as DN to DK , and the rectangle GO to the rectangle GM as HO to HM , CN is to CK as GO to GM ; therefore the rectangle AI is to the rectangle CK as the rectangle EL to the rectangle GM .

COR.

COR. If AB be to CD as EF to GH , the square upon AB will be to the square upon CD as the square upon EF to the square upon GH ; for, if any of the corresponding terms, as AB , BI , be equal to each other, the rectangle AI becomes the square upon AB .

PROP. VI.

The square of the semi-ordinate to the axis, in the ellipse and hyperbola, is to the rectangle under the abscissæ as the square of the conjugate axis is to the square of the transverse axis.

THROUGH the point G draw GVW in the ellipse, and $VGWV$ in the two hyperbolas parallel to the transverse axis AM . Then, because the lines KAG , QNq , gMk are parallel, $QR : KG :: gR : gG :: VW : GW :: NM : AM$, & $Rq : gk :: GR : Gg :: GV : GW :: AN : AM$; therefore, Lem. 1. the rectangle QRq is to the rectangle KG, gk as the rectangle ANM to the square of AM . But it may be proved, in the same manner as in the last proposition, that the rectangle QRq is equal to the square of PN ; and GK being equal to twice SA , and gk equal to twice SM , Cor. 1. Prop. 2. the rectangle KG, gk is equal to four times the rectangle ASM ; or to four times the square of BC , Def. 11. which is equal to the square of Bb ; therefore the square of PN is to the square of Bb as the rectangle ANM to the square of AM ; and alternately, the square of PN is to the rectangle ANM as the square of Bb to the square of AM .

FIG. 4.
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COR. 1. Because AM , Bb are bisected in C , the square of PN is to the rectangle ANM as the square of BC to the square of AC .

COR. 2. The axes being constant, the square of the semi-ordinate varies as the rectangle under the abscissæ.

COR. 3. The conjugate axis in the ellipse is terminated by the curve: for, when the ordinate passes through the center, the rectangle under the abscissæ is equal to the square of half the transverse axis; and therefore the square of the semi-ordinate is equal to the square of half the conjugate axis, and the ordinate is equal to the conjugate axis.

COR. 4. The two hyperbolas recede from the axis without limit: for the two abscissæ increase without limit; and therefore the square of the semi-ordinate will increase without limit.

COR. 5. Those ordinates which are at equal distances from the center of the ellipse and the two hyperbolas are equal; and those which are nearer to the center are greater in the ellipse, and less in the hyperbola, than those which are more remote: for the abscissæ which are at equal distances from the center are equal; and the rectangle ANM being equal to the difference of the squares of CM , CN , it will increase in the ellipse, and decrease in the hyperbola, as CN decreases, that is, the nearer PN is to the center.

COR. 6. Any line, which is drawn parallel to the axis of the hyperbola, will cut each of the opposite curves only in one point; for, if it be supposed to cut either of the curves in more points than one, the ordinates which are drawn through the points of intersection would be equal, when the distances from the center are unequal, which is impossible.

PROP.

P R O P. VII.

The latus rectum of the ellipse and hyperbola is a third proportional to the transverse and conjugate axes.

FOR the square of AC is to the square of BC , FIG. 4.
5.
Cor. 1. Prop. 6. as the rectangle ASM , or the square of BC , is to the square of SL ; therefore AC is to BC as BC to SL , and $2AC$ to $2BC$ as $2BC$ to $2SL$, that is, AM to Bb as Bb to LT .

COR. Hence the rectangle under the abscissæ is to the square of the semi-ordinate as the transverse axis to the latus rectum: for AM is to LT as the square of AM to the square of Bb , that is, as the rectangle ANM to the square of PN .

P R O P. VIII.

The square of half the conjugate axis, in the ellipse and hyperbola, is equal to the difference of the squares of half the transverse axis and the distance of the focus from the center.

BECAUSE AM is bisected in C , the rectangle FIG. 7.
8.
 ASM is equal to the difference of the squares of AC and SC ; but the rectangle ASM , Def. 11. is equal to the square of BC ; therefore the square of BC is equal to the difference of the squares of AC and SC .

COR. 1. If a line SB be drawn from the focus of the ellipse to the vertex of the conjugate axis, it FIG. 7.
will

COR. 1. Because AM , Bb are bisected in C , the square of PN is to the rectangle ANM as the square of BC to the square of AC .

COR. 2. The axes being constant, the square of the semi-ordinate varies as the rectangle under the abscissæ.

COR. 3. The conjugate axis in the ellipse is terminated by the curve: for, when the ordinate passes through the center, the rectangle under the abscissæ is equal to the square of half the transverse axis; and therefore the square of the semi-ordinate is equal to the square of half the conjugate axis, and the ordinate is equal to the conjugate axis.

COR. 4. The two hyperbolas recede from the axis without limit: for the two abscissæ increase without limit; and therefore the square of the semi-ordinate will increase without limit.

COR. 5. Those ordinates which are at equal distances from the center of the ellipse and the two hyperbolas are equal; and those which are nearer to the center are greater in the ellipse, and less in the hyperbola, than those which are more remote: for the abscissæ which are at equal distances from the center are equal; and the rectangle ANM being equal to the difference of the squares of CM , CN , it will increase in the ellipse, and decrease in the hyperbola, as CN decreases, that is, the nearer PN is to the center.

COR. 6. Any line, which is drawn parallel to the axis of the hyperbola, will cut each of the opposite curves only in one point; for, if it be supposed to cut either of the curves in more points than one, the ordinates which are drawn through the points of intersection would be equal, when the distances from the center are unequal, which is impossible.

P R O P.

P R O P. VII.

The latus rectum of the ellipse and hyperbola is a third proportional to the transverse and conjugate axes.

FOR the square of AC is to the square of BC , FIG. 4.
 Cor. 1. Prop. 6. as the rectangle ASM , or 5.
 the square of BC , is to the square of SL ; therefore
 AC is to BC as BC to SL , and $2AC$ to $2BC$ as
 $2BC$ to $2SL$, that is, AM to Bb as Bb to LT .

COR. Hence the rectangle under the abscissæ is to the square of the semi-ordinate as the transverse axis to the latus rectum: for AM is to LT as the square of AM to the square of Bb , that is, as the rectangle ANM to the square of PN .

P R O P. VIII.

The square of half the conjugate axis, in the ellipse and hyperbola, is equal to the difference of the squares of half the transverse axis and the distance of the focus from the center.

BECAUSE AM is bisected in C , the rectangle FIG. 7.
 ASM is equal to the difference of the squares of 8.
 AC and SC ; but the rectangle ASM , Def. 11. is equal
 to the square of BC ; therefore the square of BC is
 equal to the difference of the squares of AC and SC .

COR. 1. If a line SB be drawn from the focus FIG. 7.
 of the ellipse to the vertex of the conjugate axis, it
 will

will be equal to half the transverse axis: for, the square of BC being equal to the difference of the squares of AC , SC , the square of AC will be equal to the sum of the squares of BC , SC , or to the square of SB ; therefore SB is equal to AC .

FIG. 8. COR. 2. The distance between the vertices of the two axes, in the hyperbola, is equal to the distance of the focus from the center: for, the square of BC being equal to the difference of the squares of SC and AC , the square of SC will be equal to the sum of the squares of AC , BC , or to the square of AB ; therefore AB is equal to SC .

P R O P. IX.

The conjugate axis bisects all lines drawn parallel to the transverse axis, which are terminated by the ellipse, and by the opposite hyperbolas. Those lines which are equally distant from the center are equal; and those which are nearer to the center are greater in the ellipse, and less in the hyperbola, than those which are more remote.

FIG. 9, 10. TAKE CN any distance from the center, between C and A in the ellipse, and in CA produced in the hyperbola. Take CR , on the other side of the center, equal to CN ; and through the points N , R draw the ordinates PNp , QRq ; join PQ , pq , and let them meet the conjugate axis in z and r . Then, because the ordinates Pp , Qq are equal, Cor. 5. Prop. 6. and they are bisected in N and R , the lines PQ , NR , pq are equal and parallel; and because Pn , pr are equal to NC , and Qn ,
 qr

qr equal to RC , or NC , PQ ; pq are bisected in n and r ; and they are at equal distances from the center, because Cn , Cr are equal to PN , Np . Lastly, as Cn decreases PN decreases, and therefore CN , Cor. 5. Prop. 6. increases in the ellipse, and decreases in the hyperbola; but Pn is equal to CN , Pn therefore increases in the former, and decreases in the latter, as Cn , its distance from C , decreases.

COR. 1. The conjugate axis divides the ellipse into two equal and similar parts: the two opposite hyperbolas are equal and similar: and the ellipse and hyperbola have each of them another focus and directrix, which have the same properties as the former. Take CH , on the other side of the center, equal to CS , and Cd equal to CD : through d draw xde perpendicular to cd , and let it meet the lines PQ , pq in e , x , and join HQ . Then if the whole figure $nQMqr$ be turned round upon the axis Bb , and placed upon $nPApr$, nQ , rq will coincide with nP , rp , and all the points Q , q in the curve QMq with all the points P , p in PAp . The straight line xde will also coincide with XDE , the point H with S , and the lines HQ , Qe with the lines SP , PE ; therefore HQ is always to Qe in the same ratio that SP is to PE .

COR. 2. Suppose the line EPQ , Fig. 9. which is always parallel to DAM , to move from the center towards B : then, when Cn becomes equal to CB , the points P , Q will coincide in the point B ; and the line EPQ , which meets the curve only in one point, will be a tangent to the ellipse; and therefore the ordinates to the conjugate axis are parallel to the tangent at its vertex,

FIG. 9.

PROP.

P R O P. X.

The square of the semi-ordinate to the conjugate axis, in the Ellipse, is to the rectangle under the abscissæ as the square of the transverse axis is to the square of the conjugate axis.

FIG. 9.

FOR PnQ being parallel to AM , it is perpendicular to BCb , and it is bisected in n , by the last Prop. it is therefore an ordinate to Bb : and the square of PN , or Cn , is to the square of BC as the rectangle ANM to the square of AC , Prop. 6. therefore the difference of the squares of BC , Cn is to the square of BC as the difference of the square of AC and the rectangle ANM is to the square of AC ; but the difference of the squares of BC , Cn is equal to the rectangle Bnb , and the difference of the square of AC and the rectangle ANM is equal to the square of CN or Pn ; therefore the rectangle Bnb is to the square of CB as the square of Pn to the square of AC ; and alternately, and by inversion, we have the square of Pn to the rectangle Bnb as the square of AC to the square of BC , or as the square AM to the square of Bb .

P R O P. XI.

The transverse axis, in the Ellipse and Hyperbola, is to the distance between the directrices in the determining ratio.

FOR

FOR SA is to AD as SM to MD , and alternately, SA is to SM as AD to MD ; therefore by composition in the ellipse, and by division in the hyperbola, AM is to SA as Dd to AD , and alternately, AM is to Dd as SA to AD ; that is, Cor. 2. Def. in the determining ratio.

FIG. 9;
10.

COR. 1. Hence, Dd and AM being bisected in C , AC is to CD in the determining ratio.

COR. 2. The distance between the foci is to the transverse axis in the determining ratio. For SM is to MD as SA to AD , or as HM to AD ; and alternately, SM is to HM as MD to AD ; and by division in the ellipse, and by composition in the hyperbola, SH is to HM as AM to AD ; and alternately, SH to AM as HM to AD , or as SA to AD .

COR. 3. Hence the distance between the foci, the transverse axis, and the distance between the directrices are continual proportionals.

COR. 4. Hence, if the transverse axis and the foci of an ellipse or hyperbola be given, the determining ratio is given.

P R O P. XII.

All the diameters of an Ellipse or Hyperbola are bisected in the center.

FROM any point P in the curve draw PC to the center, and PN perpendicular to the axis. Take Cn , on the other side of the center, equal to CN ; and through the point n draw nG parallel to NP , but on the opposite side of the axis, which produce till it meet the curve in G , and join CG . Then, because Cn is equal to CN , the semi-ordinates Gn , PN will be equal, Cor. 5. Prop. 6. and the

FIG. 7,
8.

the angles at N and n being each of them a right angle, the triangle CnG will be equal to the triangle CNP ; therefore CG is equal to CP , and the angle nCG equal to the angle NCP ; and therefore PCG is one straight line, which is bisected in C .

DEFINITION XVII.

Two conic sections, or two segments of conic sections are said to be similar, when a rectilinear figure may be inscribed in one of them, similar to any rectilinear figure which is inscribed in the other.

PROP. XIII.

If the angles contained between the focal tangents be equal, or the determining ratio be the same in two conic sections, the sections will be similar.

FIG. 3. LET there be two parabolas, two ellipses, or two
 4. hyperbolas, in which the angles contained be-
 5. tween the focal tangents are equal; then SDL , which is half of the angle TDL , will be the same in both curves, and the ratio of SL to SD , or the determining ratio will be the same. And if SP makes the same angle with the axis in both, the ratio of SP to SN will be the same; but the ratio of SP to ND is the same; therefore SP is to SD , the sum or difference of ND , SD , in the same ratio in the two sections; and alternately, SP in one is to SP in the other as SD in the former to SD in the latter, that is in a given ratio. Therefore let the two sections PAQ , paq be placed in such a man-
 u ner

ner that the foci may coincide in S , and that SA may be the axis of both: Let $PBACQ$ be any rectilinear figure inscribed in one of them; join SP , SB , SA , &c. and let them meet the other section in the points p, b, a, c, q ; and join pb, ba, ac , &c. also join pq . Then the figure $pbacq$ will be similar to $PBACQ$. For SP is to Sp as SB to Sb ; and alternately, SP is to SB as Sp to Sb , and the angle PSB is common to the two triangles PSB, pSb ; therefore the triangles are similar. In the same manner it may be proved that all the other triangles are similar, and consequently the whole rectilinear figure $pbacq$ similar to the rectilinear figure $PBACQ$, and therefore the section paq is similar to the section PAQ .

COR. 1. The determining ratio in the parabola being a ratio of equality, all parabolas are similar.

COR. 2. Two ellipses or hyperbolas are similar when the axes have the same ratio to each other in both. For SC being to CA in the determining ratio, which is the same in both, the square of AC is to the square of SC , or to the difference of the squares of AC, SC , or to the square of BC , Prop. 8. in the same ratio; and consequently AC is to BC , or AM to Bb in the same ratio in both the sections.

FIG. 4.
5.

P R O P. XIV.

If from any point in the Ellipse or Hyperbola two right lines be drawn to the foci, the sum of these lines in the Ellipse, and their difference in the Hyperbola is equal to the transverse axis.

PL. IV.
FIG. 27,
28.

LET P be any point in the ellipse or hyperbola; and let S and H be the two foci. Join PS , PH ; and through the point P draw the line EPe , Fig. 27. and $P\bar{E}e$, Fig. 28. parallel to the axis; and let it meet the two directrices in E and e . Then PE , Pe will be perpendicular to the directrices, and SP will be to PE in the determining ratio, or as HP to Pe ; and alternately, SP is to HP as PE to Pe ; therefore the sum of SP , PH , Fig. 27. and their difference, Fig. 28. is to SP as Ee , or Dd , to PE ; and alternately, the sum or difference of SP , PH is to Dd as SP to PE , or, by the eleventh proposition, as AM to Dd ; therefore the sum of the lines SP , PH in the ellipse, and their difference in the hyperbola is equal to AM , the transverse axis.

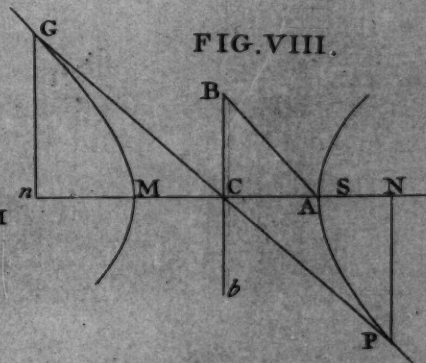
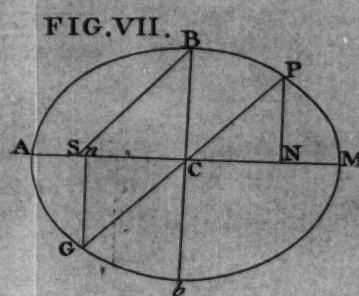
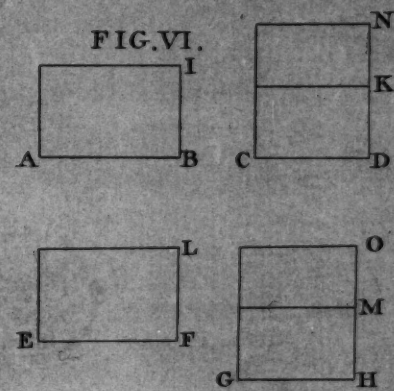
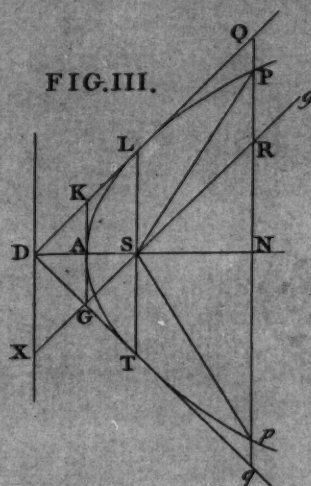
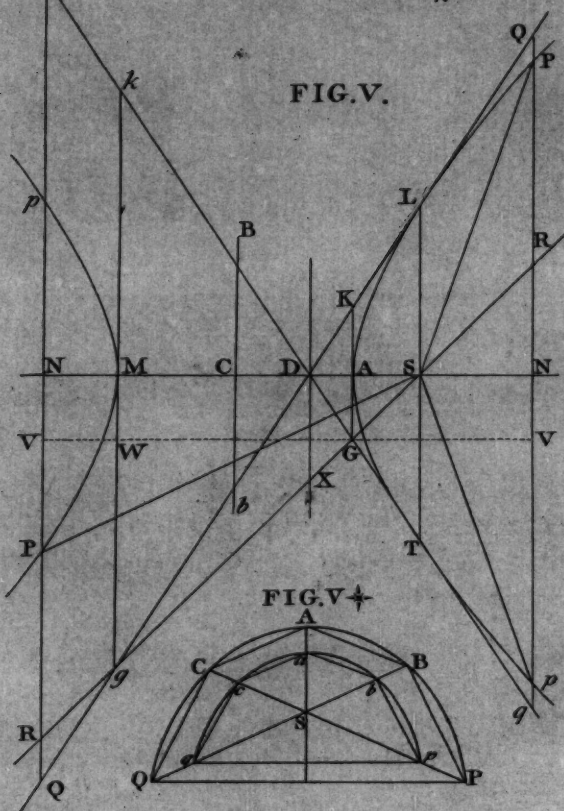
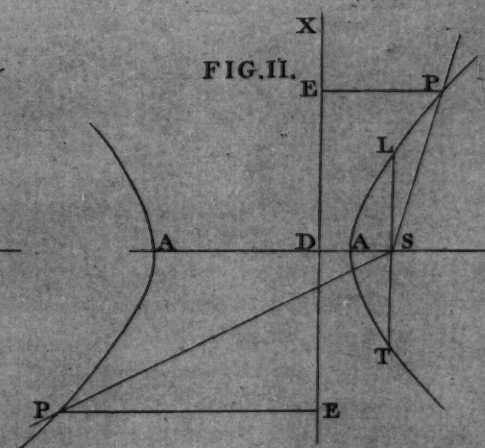
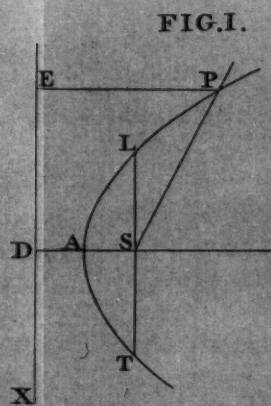
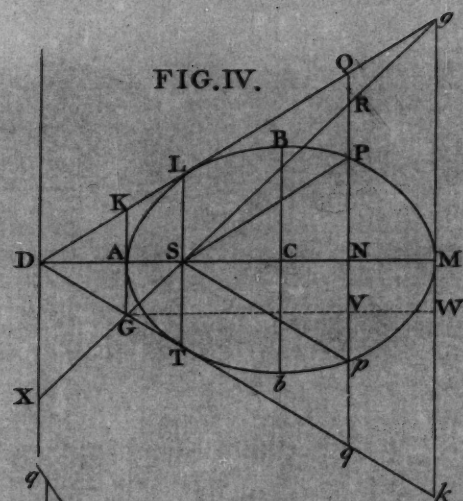
PROP. XV. PROB. II.

Two right lines being given, which bisect each other at right angles; to describe an Ellipse, or an Hyperbola, of which the given lines shall be the axes.

FIG. II.

FIRST, let the curve be an ellipse, in which case the given right lines must be unequal. Let AM , Bb be the given lines, and let Bb be less than AM . From the center B , with a radius equal to AC , describe a circle, cutting the line AM in the points S and H . Take a string equal in length to AM , and fix the extremities of it in the points S and H ; and, by means of a pin at P , let the string be stretched, and let the pin be carried round till it return to the same point. The point P will describe an ellipse, of which AM , Bb are the axes.

For



For suppose the ellipse described, of which S is the focus, A the nearest vertex, and the ratio of SH to AM the determining ratio, which may be done, by the fourth proposition; then AM will be the transverse axis, Cor. 2. Prop. 11. and Bb the conjugate axis of that ellipse, Prop. 8. and the point P will be in the ellipse in every part of its revolution. For if not, let HP meet the curve in some other point G , nearer to H , or further from it; then the sum of the lines SG, GH would be equal to AM , or to SP, PH , which is impossible.

Secondly, let AM, Bb be any two given right lines, bisecting each other at right angles in C ; and let the curve be an hyperbola. Join AB ; and from the center C take CS and CH , in AM produced both ways, equal to AB . At the point H let the end of a ruler be fixed, so that it may move freely round this point as a center; and let a string be taken, the length of which the ruler exceeds by a line equal to AM ; let one end of the string be fixed at L , and the other in the point S ; apply the string, by means of a pin at P , to the side of the ruler LH ; and let the ruler be moved about the center H , whilst the string is constantly applied, and kept close to the ruler by the pin at P . Then, the difference between the whole length of the string SPL and the ruler HL being equal to AM , the difference between HP and PS will be equal to AM ; and the point P will describe one of the opposite hyperbolas, of which AM, Bb are the axes. For suppose the hyperbola described of which S is the focus, A the vertex, and the ratio of SH to AM the determining ratio; then AM will be the transverse axis, Cor. 2. Prop. 11. and Bb the conjugate axis of that hyperbola, Prop. 8. and the point P will be in the hyperbola in every part of its revolution.

For

FIG. 12.

For if not, let the line SP meet the curve in some other point G , nearer to S , or further from it. In the first case the difference between HG , GS is equal to AM ; therefore HG is equal to AM and SG ; and HG , GP is equal to AM and PS , which is equal to HP ; therefore HP is equal to HG , GP , which is impossible. In the second case HP is equal to AM and SP ; and HP , PG equal to AM and SG , or to HG , which is also impossible. Therefore the point P must be in the curve.

PROP. XVI. PROB. III.

Two right lines being given, one of which is bisected by the other at right angles; to describe a Parabola, in which the right line bisected shall be an ordinate, and the other line the axis.

FIG. 13. **L**ET AC , Bb be the two given right lines, one of which Bb , which is perpendicular to AC , is bisected in C . Find a third proportional to AC , CB ; and produce CA to D , so that AD may be a fourth part of that third proportional; and take AS equal to AD . Through D draw DEX perpendicular to DAC ; and let a ruler, the sides of which HE , EL are perpendicular to each other, be placed in the plane CDX , so that the side EL may be applied to DX ; and take a string equal in length to the side HE , one extremity of which must be fixed at H , and the other at S ; and let part of the string be applied, by means of a pin P , to the side of the ruler HE ; and whilst the side EL moves along DX , let the string be stretched by the pin, and
con-

constantly applied to HE . Then, because the whole length of the string HPS is equal to HE , the part SP will be always equal to PE ; therefore the point P will describe a parabola, by the first definition, of which AC is the axis, S the focus, and DX the directrix; and BCb will be an ordinate of that parabola, because it is perpendicular to the axis, and CB is a mean proportional between the abscissa AC and four times AS , or the latus rectum.

P R O P. XVII.

If any point be taken within a conic section, its distance from the focus will be to its distance from the directrix in a less ratio than the determining ratio; and if the point be taken without the section, its distance from the focus will be to its distance from the directrix in a greater ratio.

Part 1. **L**ET the point L be taken any where within the section which is on the same side of the directrix with the focus, Fig. 14. or within the opposite section, Fig. 15. Join LS , and let it meet the curve in P . Draw LM, PE perpendicular to the directrix. Join SE , and let it meet the line LM in F . Then, because the triangles SLF, SPE are similar, SL is to LF as SP to PE ; but SL is to LM in a less ratio than SL to LF ; and therefore in a less ratio than the determining ratio.

Part 2. Let the point L be taken without the conic section, but on the same side of the directrix with

FIG. 14.
15.

FIG. 14.

with S . Join LS , which will cut the curve in P ; and draw LM , PE perpendicular to the directrix. Join SM , cutting the line PE in G ; and because the triangles SLM , SPG are similar, SL is to LM as SP to PG , which is a greater ratio than that of SP to PE . Secondly, let the point L be on the other side of the directrix, and let the section be an ellipse or a parabola. Join LS , cutting the directrix in N , and the curve in P . Draw LM , PE perpendicular to the directrix; and join SE , which produce till it meet ML produced in O . Then SL is to LO as SP to PE ; therefore LO is not less than SL ; and SL is greater than LM ; therefore LO must be greater than LM ; and SL is to LM in a greater ratio than SL to LO , or SP to PE .

FIG. 15.

Lastly, let the point L be taken any where between the directrix and the opposite hyperbola. Draw LE perpendicular to the directrix, which produce till it meet the opposite hyperbola in P ; and join SP . Through P draw PF parallel to the directrix, meeting SL produced in F ; and draw FM parallel to PE . Join SE , and produce it till it meet FM in G . Then it is evident that SF cannot be less than SP ; and SL is to LE as SF to FG , that is, in a greater ratio than SF to FM , or than SP to PE .

COR. Hence, if the distance of any point from the focus of a conic section be to its distance from the directrix in the determining ratio, that point is in the section; and if its distance from the focus be to its distance from the directrix in a greater, or in a less ratio, the point will be without, or within the conic section.

P R O P.

PROP. XVIII. PROB. IV.

The focus, the directrix, and the determining ratio being given; to find the points in which a straight line passing through the focus, which is given in position, meets the conic section, or opposite sections.

IF the line which passes through the focus be parallel to the directrix, it will coincide with the latus rectum, and the proposition is manifest. If the line SQ be not parallel to the directrix, let it meet it in some point Q . Take QH and QG in the directrix, on opposite sides of Q , each of them equal to QS . Draw the latus rectum LST ; and join LG , LH ; and produce LH till it meet QS produced, if possible, in p . The points P , p , in which LG and LH meet the line QS , will be in the conic section. For draw PE , pe perpendicular to the directrix; and because the triangles SPL , PQG are similar, as also the triangles PQE , SQD , SP is to SL as PQ to QG , or QS , or as PE to SD ; and alternately, SP is to PE as SL to SD , that is, in the determining ratio; and therefore, Cor. Prop. 17. the point P is in the curve. And because the triangles pSL , pQH are similar, as also the triangles pQe , SQD , Sp is to SL as Qp to QH , or QS , or as pe to SD ; and alternately, Sp is to pe as SL to SD ; therefore p is also in the curve; and the line QS meets the curve in the points P and p .

COR. 1. Every line which passes through the focus of the parabola will meet the curve in two points, except that which is perpendicular to the directrix. For it is evident that it will meet the

D

curve

FIG. 16,
17.

FIG. 16] curve in some point P , between the latus rectum and the directrix, in all the conic sections; and when $\mathcal{Q}$ coincides with D , $S\mathcal{Q}$ will be equal to SL ; therefore SL is equal to $\mathcal{Q}H$, and the lines $\mathcal{Q}S$, HL are parallel; but if $\mathcal{Q}$ does not coincide with D , $S\mathcal{Q}$ will be greater than SD ; and therefore $\mathcal{Q}H$ greater than SL , and the lines $\mathcal{Q}S$, HL will meet in some point p , which is in the curve, by the proposition.

COR. 2. Every line which passes through the focus of an ellipse will meet the curve in two points.

COR. 3. If a line passing through the focus of an hyperbola be inclined to the directrix at such an angle, that the radius is to the sine of it in the determining ratio, it will meet the curve only in one point: if it be inclined at a less angle, it will meet the same hyperbola in two points; and if it be inclined at a greater angle, it will meet each of the opposite curves in one point.

FIG. 16, 17. First, let the radius be to the sine of the angle $S\mathcal{Q}D$ in the determining ratio; then $S\mathcal{Q}$ will be to SD as SL to SD ; therefore $S\mathcal{Q}$, or $\mathcal{Q}H$, will be equal to SL , and the lines $S\mathcal{Q}$, LH will be parallel.

FIG. 16. Secondly, let the line $S\mathcal{Q}$ be inclined to the directrix at a less angle; then $S\mathcal{Q}$ is to SD in a greater ratio than SL to SD ; therefore $S\mathcal{Q}$, or $\mathcal{Q}H$, is greater

FIG. 17. than SL ; and $\mathcal{Q}S$, HL will meet in some point p in the direction $\mathcal{Q}S$. Lastly, let the angle $S\mathcal{Q}D$ be greater; then $S\mathcal{Q}$ is to SD in a less ratio than SL to SD ; therefore $S\mathcal{Q}$, or $\mathcal{Q}H$, is less than SL ; and the lines $S\mathcal{Q}$, LH will meet in the opposite curve.

DEFINITION

FIG. IX.

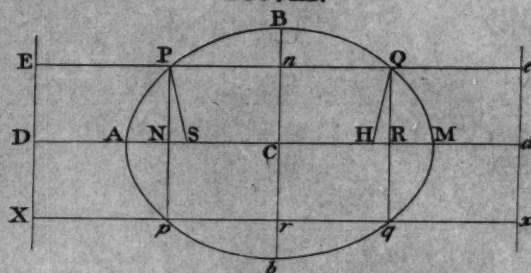


FIG. XI.

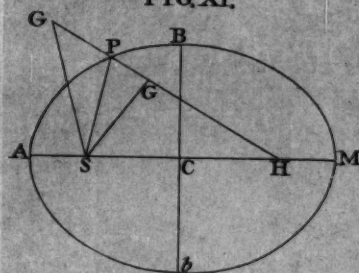


FIG. X.

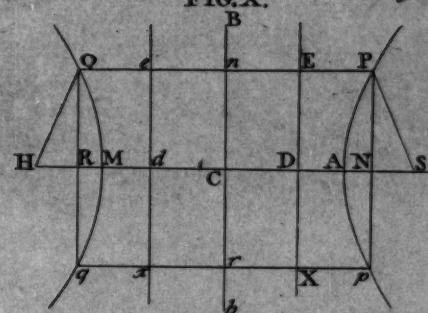


FIG. XVI.

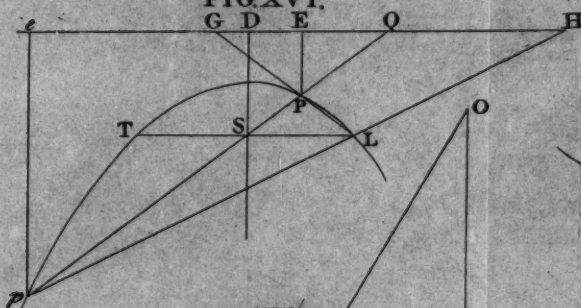


FIG. XVII.

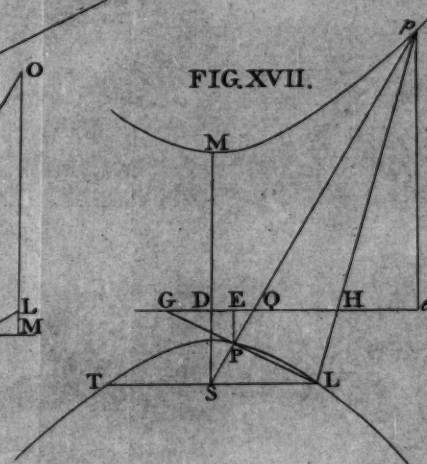


FIG. XIII.

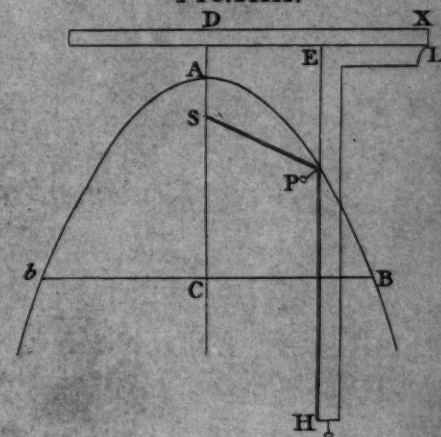


FIG. XIV.

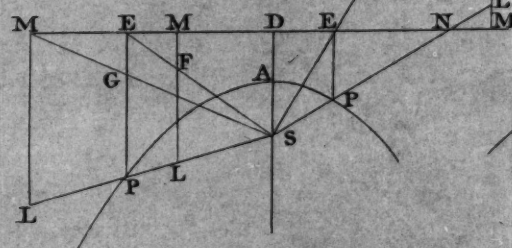


FIG. XII.

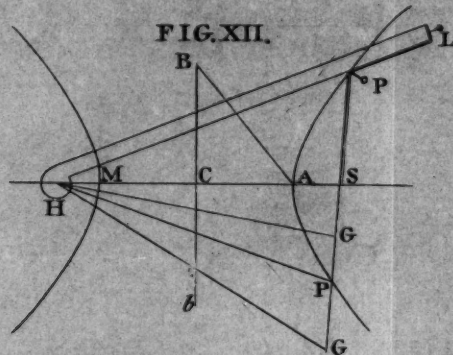
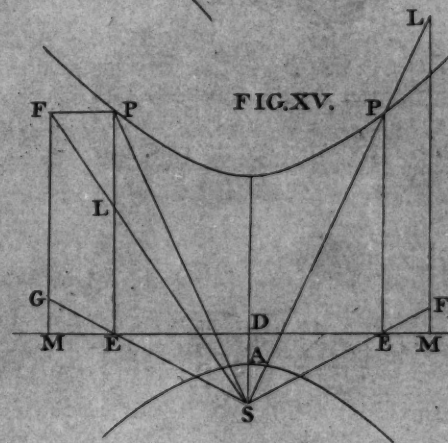


FIG. XV.



DEFINITION XVIII.

If from the center C , at the distance CA , half the transverse axis, a circle be described, cutting the directrix of the hyperbola in the points H, h , and lines be drawn from the center through the points of intersection, these lines are called the Asymptotes. FIG. 18.

PROP. XIX.

If lines be drawn from the focus to the points in which the asymptotes cut the directrix, they will be perpendicular to the asymptotes: and the angle contained between the asymptote and the directrix is such, that the radius is to the sine of it in the determining ratio.

LET the asymptotes Ca, CG cut the directrix in the points H, h ; and join SH, Sh . Then CS is to CA as CA to CD , Cor. 3. Prop. 11. but CH is equal to CA , therefore CS is to CH as CH to CD , and the angle SCH is common to the two triangles SHC, CDH ; therefore the triangles are similar, and the angle SHC is equal to the angle CDH , which is a right angle. In the same manner it may be proved that the angle ShC is a right angle. Secondly, CA , or CH , is to CD in the determining ratio; and, CH being made radius, CD is the sine of the angle CHD . FIG. 18.

COR. 1. Hence, if a line be drawn through the focus parallel to an asymptote, it will cut the curve only in one point.

FIG. 19. COR. 2. If a line PG be drawn from any point P in the hyperbola, or in the opposite curve, parallel to the asymptote, meeting the directrix in G , PG will be equal to PS . From P draw PE perpendicular to the directrix; and because the angle PGE is equal to the angle CHD , PG is to PE in the determining ratio, or as SP to PE ; therefore PG is equal to SP .

COR. 3. Hence, if the focus, the directrix, and the position of the asymptote be given, the hyperbola may be described, by means of a ruler and string, in the same manner as the parabola is described, Prop. 16. provided the sides of the ruler be inclined to each other in the same angle as the asymptote and directrix.

P R O P. XX.

The asymptotes never meet the curve: but any other line drawn parallel to an asymptote will meet one of the hyperbolas.

FIG. 19. FOR if it be possible, let the asymptote meet the curve in the point R . Join RS , and draw RN perpendicular to the directrix. Then, by the preceding proposition, HR is to RN in the determining ratio, or as RS to RN ; therefore RS is equal to RH , and the angle RSH is equal to the angle RHS ; which is impossible, the angle RHS being a right angle. In the same manner it may be proved that it cannot meet the opposite curve.

Let any other line GP be drawn parallel to the asymptote; and first let it be nearer to the focus. Join SG , and produce it till it meet the asymptote CH in I ; then the angle SGP is equal to the angle SIH ,

SIH , which is less than the angle SHR , a right angle; therefore, the angle SGP being less than a right angle, if the angle GSP be made equal to SGP , SP , GP will meet, when produced, somewhere in P , which is a point in the curve. For draw PE perpendicular to the directrix; and the angle PGE being equal to the angle CHD , PG is to PE in the determining ratio; therefore SP is to PE in the same ratio, and P , Cor. Prop. 17. is in the hyperbola. Secondly, let gp be drawn parallel to the asymptote, at a greater distance from the focus. Join Sg , cutting the asymptote in i ; then the angle Sgp is equal to the angle SiH , which is less than the angle SHi , a right angle: if therefore the angle gSp be made equal to Sgp , the lines Sp , gp will meet, when produced, in some point p , which is in the opposite hyperbola; for the angle pge being equal to CHD , pg is to pe in the determining ratio; and therefore Sp is to pe in the same ratio, and the point p , Cor. Prop. 17. is in the curve.

COR. Hence it is evident, that if any line be drawn through the center of an hyperbola within the angle contained between the asymptotes it will meet both the curves.

P R O P. XXI.

The distance between the focus and the point in which the asymptote cuts the directrix, as also the tangent at the vertex, intercepted between the vertex and asymptote, are each of them equal to half the conjugate axis.

F O R

FIG. 18. **F**OR the square of SH is equal to the difference of the squares of SC , CH , or to the difference of the squares of SC , CA ; which, Prop. 8. is equal to the square of BC ; therefore SH is equal to BC . Secondly, because CH is equal to CA , and the angles SHC , CAa , are right angles, and the angle SCH is common to the two triangles SHC , CAa , the triangles are equal, and Aa is equal to SH , which was proved equal to BC .

COR. If Aa and AG be taken in the tangent GAA each of them equal to CB , and the lines Ca , CG be drawn from the center, the position of the asymptotes will be determined.

DEFINITIONS.

FIG. 18. XIX. If AM be the transverse axis, and Bb the conjugate axis of any two opposite hyperbolas, and two other hyperbolas be described, of which the transverse axis is Bb , and the conjugate axis AM , these hyperbolas are said to be conjugate to the former.

XX. When the two axes are equal the hyperbolas are said to be equilateral.

PROP. XXII.

The asymptotes are diagonals of the rectangle which is made by drawing tangents through the vertices of the four hyperbolas.

FIG. 18. **L**ET the tangents GAA , IMi , which are drawn through the vertices of the transverse axis, meet the asymptotes in G , a , and I , i . Join IB , GB , as also ab , ib . Then, because CM is equal to CA , and
 o the

the angles at A and M are right angles, and the angles at C vertical, the triangle CMI will be equal to the triangle CAa , and MI equal to Aa , which is equal to CB , by the preceding proposition. In the same manner it may be proved that Mi is equal to Cb , or to CB ; therefore IB, BG are equal and parallel to MC, CA , the angles IBC, GBC are each of them a right angle, and IBG is one straight line, which is equal and parallel to MA . For the same reason iba is one straight line, which is equal and parallel to MA ; and because the lines IBG, iba are perpendicular to the axis BCb , they are tangents to the conjugate hyperbolas, and $IGai$ is a rectangle, of which the asymptotes Ia, Gi are the diagonals.

COR. 1. The asymptotes GCI, ICa are also asymptotes to the conjugate hyperbolas. For BI and BG are each of them equal to CA , which is the semi-conjugate axis to the hyperbolas LBR, lbr .

COR. 2. If the hyperbolas be equilateral, the asymptotes will be perpendicular to each other; for CA being equal to AG , or to Aa , each of the angles ACG, ACa will be half a right angle; and therefore the angle GCa will be a right angle.

P R O P. XXIII.

If a right line Pp , which cuts a conic section FIG. 20.
or opposite sections in two points P, p , 21.
meets the directrix in H , and a right line 22.
 HST be drawn through the focus, and $SP,$
 Sp be joined; the angle PSH will be equal
to the angle pST .

DRAW

DRAW pT parallel to PS , and let it meet HS in T ; and draw PE , pe perpendicular to the directrix. Then the triangles HPE , Hpe will be similar, as also the triangles HSP , HTp ; and SP is to PE as Sp to pe ; and alternately, SP is to Sp as PE to pe , as HP to Hp , or as SP to Tp ; therefore Sp is equal to Tp , and the angle pST is equal to the angle pTS , which is equal to the angle PSH .

FIG. 20, COR. 1. When P and p coincide, or when HP
22. becomes a tangent to the conic section, SP will coincide with Sp , and each of the angles PSH , pST will be a right angle.

FIG. 23, COR. 2. Hence, if a line SP be drawn from the
24. focus to any point P in a conic section, and SH be
25. drawn perpendicular to SP , meeting the directrix in H , and HP be joined, it will touch the conic section in the point P .

FIG. 20, COR. 3. Let the line HP , which meets the di-
21. rectrix in H , cut a conic section in any point P ;
22. join SP , draw HST through the focus, and make the angle TSp equal to the angle HSP ; then if HP , Sp be produced till they meet in p , the point p will be in the conic section, or in the opposite section. For the angle TSp is equal to the angle HSP , or to the angle STp , and Tp is equal to Sp ; and Tp , or Sp , is to SP as pH to PH , or as pe to PE ; and alternately, Sp is to pe as SP to PE ; therefore p is a point in the conic section, Cor. Prop. 17.

COR. 4. Any line drawn parallel to an asymptote will meet the hyperbola, or the opposite hyperbola, only in one point.

FIG. 20, It is evident that it will only meet one of the hy-
22. perbolas, from Prop. 20. Therefore let HP be drawn parallel to an asymptote, meeting the hyperbola in P ; and if it be possible, let it meet the

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same curve in some other point p ; and join SP, Sp . Because HP is equal to SP , Cor. 2. Prop. 19. the angle SHP is equal to the angle PSH , or to the angle pST ; which is impossible, pST being the exterior angle of the triangle pHS .

COR. 5. It is evident, from this proposition, that a straight line cannot meet a conic section in more points than two.

P R O P. XXIV.

If two tangents be drawn at the extremities of any line passing through the focus of a conic section, which is terminated both ways by the curve, or by the opposite curves, they will meet in the directrix; and they will contain a right angle in the parabola, an acute angle in the ellipse, and an obtuse angle, or an acute angle in the hyperbola, according as the line which passes through the focus is terminated by the same, or by the opposite curves.

LET PSp be any line passing through the focus, which is terminated by the curve, or by the opposite curves in P, p . From S draw SH perpendicular to PSp , meeting the directrix in H , and join HP, Hp , which will touch the curve in the points P, p , Cor. 2. Prop. 23. Draw PE, pe perpendicular to the directrix; and SP, PE will be the sines of the angles SHP, PHE , HP being made radius, and Sp, pe the sines of the angles SHp, pHe , Hp being radius. But SP is equal to PE in the

FIG. 23;
24,
25,
26.

E

para-

- FIG. 24. parabola, and Sp is equal to pe ; therefore the angle SHP is equal to the angle PHE , and the angle SHp equal to the angle pHe ; and the angle PHp is equal to the sum of the angles PHE , pHe ; therefore it is a right angle. If the section be an ellipse, SP will be less than PE ; therefore the angle SHP will be less than the angle PHE , and the angle SHp less than the angle pHe , and the whole angle PHp less than the sum of the angles PHE , pHe , and consequently less than a right angle. If the line PSp be terminated by the same hyperbola, SP will be greater than PE , and Sp greater than pe ; therefore the angle PHp will be greater than the sum of the angles PHE , pHe , and consequently greater than a right angle. But if the line SPp be terminated by the opposite curves, the angle SHp , in the right angled triangle HSp , is less than a right angle; and therefore the angle PHp is less than a right angle.

P R O P. XXV.

If a tangent be drawn to any point in the Parabola, it will bisect the angle contained between two right lines drawn from the point of contact, one to the focus, and the other perpendicular to the directrix.

- FIG. 24. **L**ET the line PH , which touches the parabola in any point P , meet the directrix in H . Join SP , SH ; and draw PE perpendicular to the directrix. The angle SPE is bisected by the line PH . For the angle PHS is equal to the angle PHE , the angles PSH , PEH are right angles, and PH is common to the two triangles PSH , PEH ; therefore

fore the triangles are equal; and the angle SPH is equal to the angle EPH .

COR. 1. Hence, if the right line PH bisects the angle SPE , it will be a tangent to the parabola in the point P .

COR. 2. If the tangent be produced till it meet the axis in T , the segment of the axis intercepted between the focus and the tangent will be equal to the distance of the focus from the point of contact. For the angle STP is equal to the alternate angle TPE , which is equal to the angle SPT ; therefore ST is equal to SP .

PL. VII.

FIG. 51.

P R O P. XXVI.

If two tangents PH, pH be drawn at the extremities of any line which passes the focus of the Parabola, and a right line HI be drawn from the point of concurrence parallel to the axis, it will bisect the line Pp in I : and HI will be bisected by the curve in the point A .

FIG. 24.

FOR the angle IHP is equal to the alternate angle HPE , which is equal to the angle HPI , by the preceding proposition; therefore IP is equal to IH ; and for the same reason Ip is equal to Ih ; and therefore IP is equal to Ip . Secondly, SA being equal to AH , the angle ASH is equal to AHS ; but the angle SHI and SIH are together equal to a right angle, and therefore equal to ASH and ASI ; and if from these be taken the equal angles SHI, ASH , the remaining angles AIS, ASI will be equal, and AI will be equal to AS , which is equal to AH .

P R O P. XXVII.

If a tangent be drawn to any point in an Ellipse or an Hyperbola, and two lines be drawn from the point of contact to the foci; the angles contained between each of these lines and the tangent are equal.

FIG. 27,
28. **L**ET the line PT touch the ellipse or hyperbola in any point P , and let it meet the directrices in T and t . Through P draw EPe , Fig. 27. and PEe , Fig. 28. parallel to the axis AM , meeting the directrices in E and e , which will be perpendicular to the directrices. Draw PS , PH to the foci, and join ST , Ht . Because the triangles TPE , tPe are similar, PE is to PT as Pe to Pt , and

SP is to PE as HP to Pe ; therefore SP is to PT as HP to Pt , and the angles PST , PHt are right angles, Cor. 1. Prop. 23. therefore the triangles SPT , HPt are similar, and the angle SPT is equal to the angle HPt .

COR. 1. If a line TP be drawn through any point P in the ellipse or hyperbola, bisecting the angle SPH in the latter, and its supplement in the former, it will touch the curve in the point P .

COR. 2. If a line PR be drawn from any point P perpendicular to the tangent, meeting the axis in R ; it will bisect the angle contained between the lines which are drawn from the point of contact to the foci in the ellipse, and the angle which is contained between one of these lines and the other produced in the hyperbola. For the angle RPT being equal to the angle RPt , Fig. 27. and equal to RPW , Fig. 28. if from each of these be taken SPT ,
 HPt ,

FIG. XVIII.

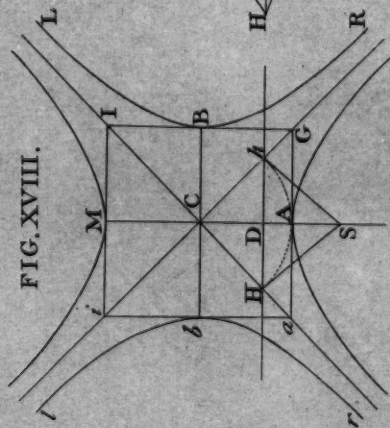


FIG. XXII.

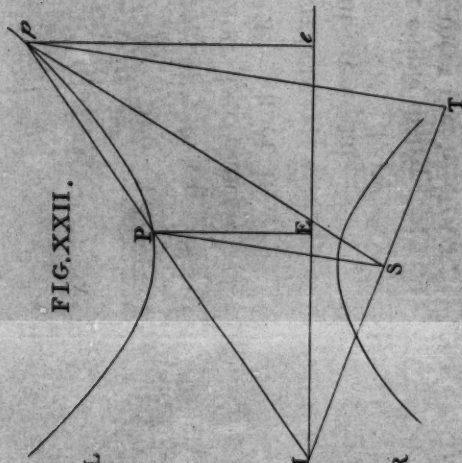


FIG. XXI.

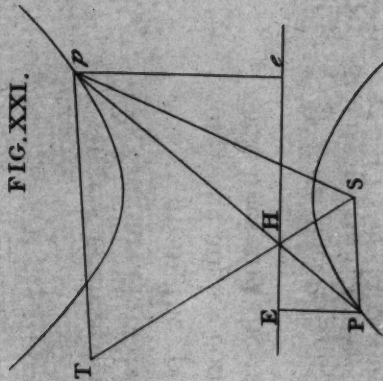


FIG. XIX.

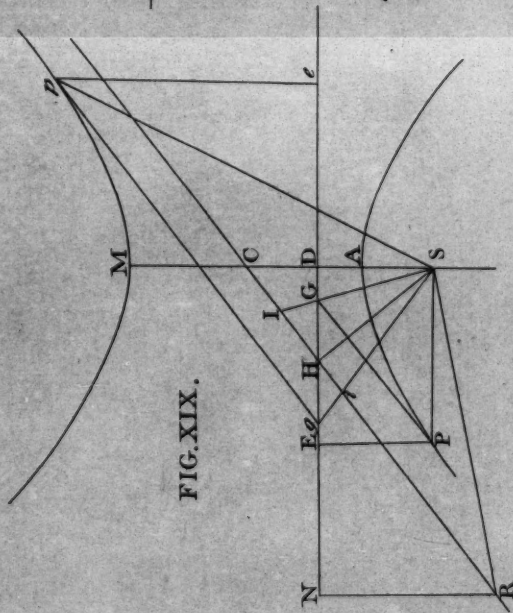


FIG. XX.

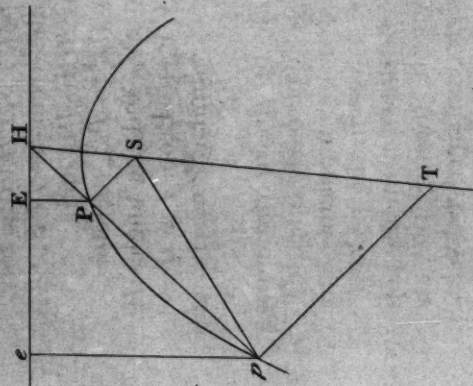


FIG. XXIII.

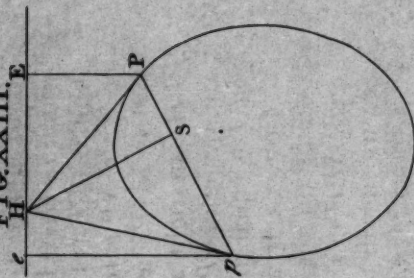


FIG. XXVI.

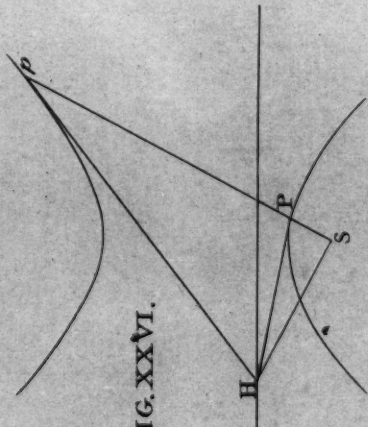


FIG. XXV.



FIG. XXIV.

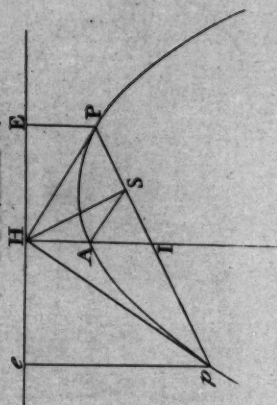


FIG. XVIII.

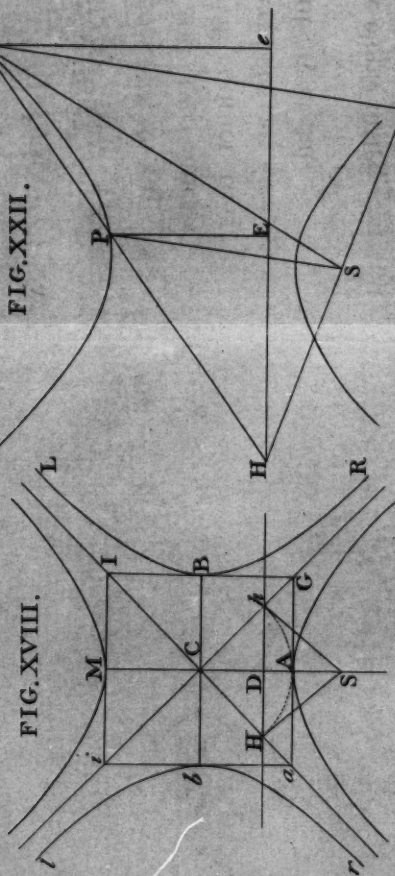


FIG. XXII.

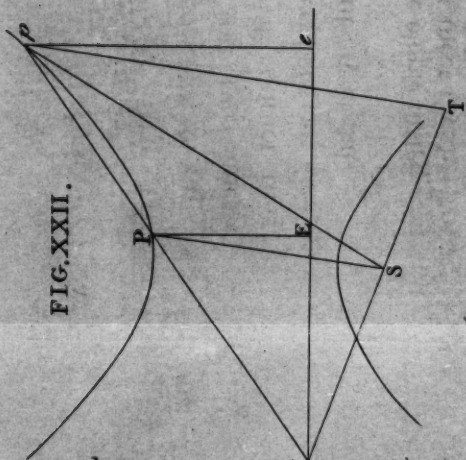


FIG. XXI.

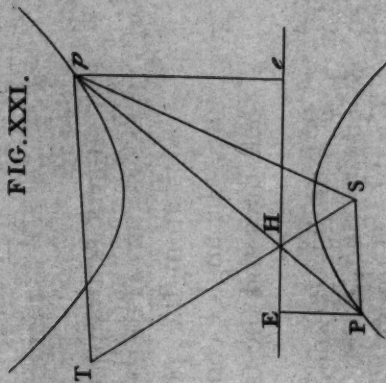


FIG. XX.

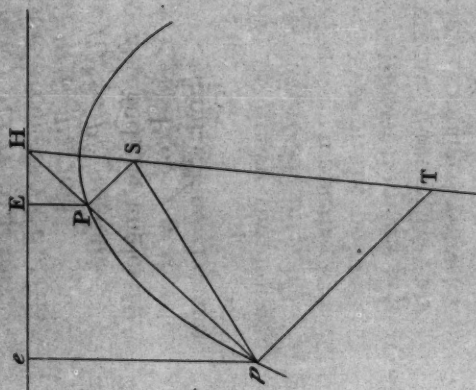


FIG. XXIII.

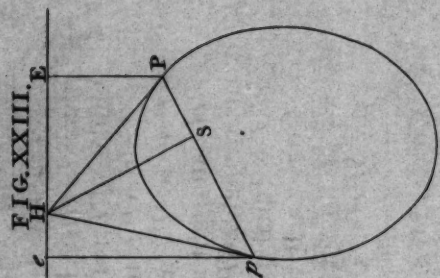


FIG. XIX.

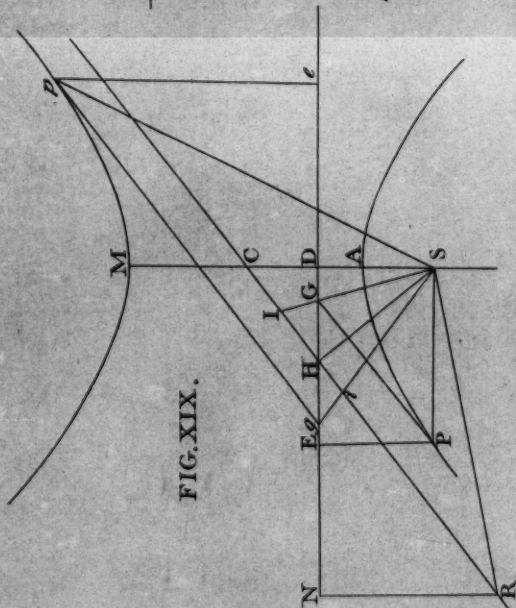


FIG. XXIV.

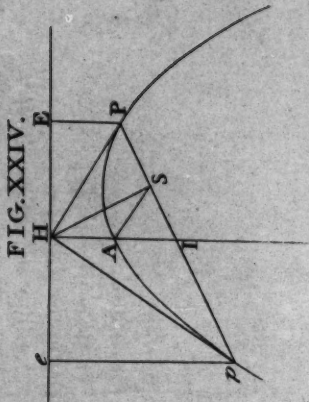


FIG. XXV.

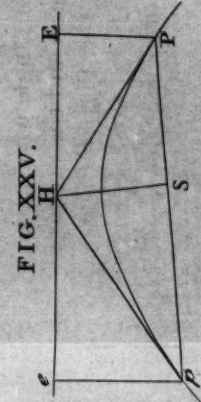
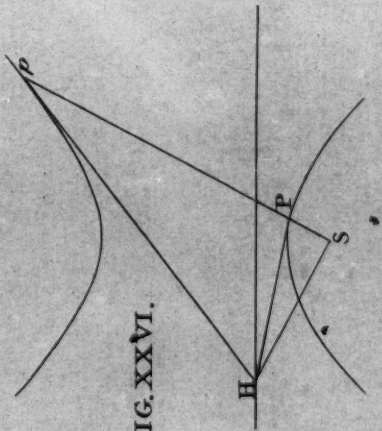


FIG. XXVI.



HPt, Fig. 27. and *SPt*, *hPW*, Fig. 28. the remaining angles *RPS*, *RPH* in the ellipse, and *RPS*, *RPk* in the hyperbola will be equal.

P R O P. XXVIII.

The tangents at the vertices of any diameter of an Ellipse or an Hyperbola are parallel.

FIG. 29,
30.

LET *PCG* be any diameter of the ellipse or hyperbola. Draw the tangents *PQ*, *GR*; and join *SP*, *PH* and *SG*, *GH*. Then, because *SC* is equal to *CH*, *CP* equal to *CG*, and the angles at *C* are vertical, the triangles *SCP*, *HCG* are equal, and the angle *SPC* is equal to the angle *HGC*, and *SP* is equal and parallel to *GH*; therefore *PH* is equal and parallel to *SG*, and *SPHG* is a parallelogram; therefore the angle *SPH* is equal to the angle *SGH*; and the halves of these angles, Fig. 30. and the halves of their supplements, Fig. 29. will be equal, that is, the angle *SPQ* equal to the angle *HGR*; and if these be added to the equal angles *SPC*, *CGH* in the ellipse, and subtracted from them in the hyperbola, *CPQ* will be equal to *CGR*; and therefore *PQ* is parallel to *GR*.

COR. If two tangents be drawn at the vertices of any other diameter of the ellipse, or of any diameter of the conjugate hyperbolas, which are produced to meet the tangents *PQ*, *GR*, a parallelogram will be formed, which is circumscribed about the ellipse, and inscribed in the four hyperbolas.

DEFINITIONS

DEFINITIONS.

FIG. 27. XXI. The right line PR , which is drawn from
28. the point of contact perpendicular to the tangent, intercepted between the tangent and the axis of a conic section, is called a Normal.

FIG. 28. XXII. The segment of the axis NR , which is intercepted between the ordinate and the normal, is called a Subnormal.

XXIII. If a right line be drawn through any point in the diameter of a conic section parallel to the tangent at its vertex, which is terminated both ways by the curve, it is called an Ordinate to that diameter.

XXIV. The segment of any diameter of a conic section, which is intercepted between an ordinate and the vertex, is called an Abscissa.

XXV. A diameter which is parallel to the tangent at the vertex of any diameter of the ellipse or hyperbola, is called a Conjugate Diameter.

XXVI. A line which is a third proportional to any diameter of the ellipse or hyperbola and its conjugate, is called a Parameter to that diameter.

XXVII. If a line be drawn through the focus of a parabola parallel to the ordinates of any diameter, which is terminated both ways by the curve, it is called a Parameter to that diameter.

PROP. XXIX. PROB. V.

To draw a tangent to a conic section from any given point without, which is not the center of the hyperbola.

IF

IF the given point H be in the directrix; draw HS to the focus which is nearest to the directrix; draw SP perpendicular to SH , meeting the curve in P , and join HP , which will touch the conic section in P , Cor. 1. Prop. 23. FIG. 31.
32.

If the given point be in any other situation, as at L ; join LS , and draw LX perpendicular to the directrix. Take LD to LX in the determining ratio, and from the center L , at the distances LD , describe a circle DMq ; and because LS is to LX in a greater ratio than LD to LX , Prop. 17. LS is greater than LD , and the point S is without the circle. From S draw SQ a tangent to the circle, and let it meet the directrix in H . Join LQ , and draw SP parallel to it, or perpendicular to SH . Join HL and produce it till meet SP in the point P , which is in the conic section, and the line HP touches the curve in P . For, because the triangles HQL , HSP are similar, as also the triangles LHX , PHE , SP is to PH as QL to LH , and

PH is to PE as LH to LX ; therefore SP is to PE as QL to LX , that is, in the determining ratio; therefore P is a point in the curve; and because PSH is a right angle, PH is a tangent, Cor. 1. Prop. 23.

COR. 1. Because two lines SQ , Sq may be drawn from the point S to touch the circle, two tangents LP , Lp may be drawn from the point L to the conic section.

COR. 2. The lines LP , Lp will touch the hyperbola which is on the same side of the directrix with the focus, or the opposite curve, according as SQ , Sq touch the circle on the opposite side, or on the same side of the directrix with the focus. FIG. 32.

COR. 3. If the line SQ meets the circle in the directrix, the line LHP becomes an asymptote. For

For in that case LQ and LH would coincide; and LQ would be to LX as radius to the sine of the angle LHX ; therefore LH would be inclined to the directrix at the same angle as the asymptote is; and it would never meet the curve, because SP is parallel to LQ , or LH ; therefore LH coincides with the asymptote.

P R O P. XXX.

FIG. 33, 34, 35. If two right lines Pp, Qq , which meet each other in any point L , and are inclined to the directrix at any given angles LHX, LbX , cut a conic section, or opposite sections, in the points P, p and Q, q ; the rectangles under the segments LP, Lp and LQ, Lq will be in a constant ratio to each other, wherever the point L be taken.

LET S be the nearest focus. Join HS , and produce it if necessary; also join SP, Sp . Draw LX, PE perpendicular to the directrix; and from the point L draw LT, Lt parallel to SP, Sp , meeting the line HS in T and t . Because the angle PSH , Prop. 23. is equal to the angle pST , Fig. 33 and 35. and equal to pSW , Fig. 34. the angle LTt is equal to the angle LtT , and LT is equal to Lt . From the center L , at the distance LT , or Lt , describe a circle, cutting the line HPp , in M and m . Join SL , and produce it till it meet the circle in D and d ; and because the triangles HPE, HLX are similar, as also the triangles HPS, HLT , LT is to SP as LH to PH , or as LX to PE ; and alternately, LT is to

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LX as SP to PE , that is, in the determining ratio; therefore the radius of the circle is the same when L is at the same distance from the directrix, whatever be the position of the line Pp . And because LT is parallel to PS , and Lt parallel pS ,

LP is to TS as LH is to TH , and

pL is to St as LH is to tH ; therefore

Lem. 1. the rectangle PLp is to the rectangle TSt as the square of LH is to the rectangle THt ; but the rectangle TSt is equal to the rectangle DSd , and the rectangle THt is equal to the rectangle MHm , or to the difference of the squares of LH and LM ; therefore the rectangle PLp is to the rectangle DSd as the square of LH is to the difference of the squares of LH and LM ; but LH is to LT , or LM , as PH to PS , and the square of LH is to the square of LM as the square PH is to the square of PS ; and by division, the square of LH is to the difference of the squares of LH and LM as the square of PH is to the difference of the squares of PH and PS ; which ratio depends only upon the determining ratio and the angle LHX , SP being to PH in a ratio which is compounded of the ratios of SP to PE , and PE to PH , or of the determining ratio, and the sine of the angle LHX to radius. In the same manner it may be proved, that the rectangle QLq is to the rectangle DSd in a ratio which depends only on the determining ratio and the angle LHX ; therefore the rectangle PLp is to the rectangle QLq in a constant ratio, whatever be the distance of the point L from the directrix.

COR. 1. If either of the lines Pp , Qq , or both of them become tangents to the conic section, or opposite sections, the squares of the tangents must be substituted for the rectangles PLp , QLq . For FIG. 31,
let LP touch the conic section in P . Then, QL 32.
being

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being parallel to SP , by the preceding proposition, LP is to QS as LH to QH ; and the square of LP , Lem. 1. is to the square of QS as the square of LH is to the square QH ; but the square of QS is equal to the rectangle DSd , and the square of QH is equal to the rectangle MHm ; therefore the square of LP is to the rectangle DSd as the square of LH is to the rectangle MHm , or to the difference of the squares of LH, LM , which was proved to be a constant ratio; therefore the square of LP is to the square of Lp , or to the rectangle QLq , Fig. 34. in a constant ratio.

COR. 2. If the determining ratio be that of a to b , and the sines of the angles contained between each of the lines Pp, Qq and the directrix be R and S , the radius being unity, the ratio of the rectangles PLp and QLq will be that of $b^2 - a^2 R^2$ to $b^2 - a^2 S^2$.

For SP is to PE as a to b , and

PE is to PH as R to 1; therefore

SP is to PH as aR to b , and the square of SP is to the square of PH as $a^2 R^2$ to b^2 , and the square of PH is to the difference of the squares of PH and SP as b^2 to $b^2 - a^2 R^2$: therefore

$LP \times Lp : SD \times Sd :: b^2 : b^2 - a^2 R^2$, and

$SD \times Sd : LQ \times Lq :: b^2 - a^2 S^2 : b^2$; therefore

$PL \times Lp : LQ \times Lq :: b^2 - a^2 S^2 : b^2 - a^2 R^2$.

COR. 3. If the focus, the directrix, and the determining ratio be given, and a right line be given in position; the points in which it meets the conic section, or opposite sections, may be found. If the right line passes through the focus, the points of intersection may be found by Prop. 18. and if it be parallel to the directrix, by the fourth proposition.

Therefore let the right line LH , which does not pass through the focus, meet the directrix in H .
 FIG. 33, 34, 35. Take any point L in that line; draw LS to the focus,

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cus, and LX perpendicular to the directrix. In LS , produced if necessary, take LD to LX in the determining ratio; and from the center L , at the distance LD , describe a circle; and join HS . Then it is evident, from this and the preceding proposition, that the line HS will cut the circle in two points T, t , or touch it, according as the line HL cuts the conic section, or touches it. Therefore join LT, Lt ; and draw SP, Sp parallel to LT, Lt , meeting the line HL in P, p , which are points in the section, or sections. For SP is to TL as PH is to LH , or as PE to LX ; and alternately, SP is to PE as TL to LX , that is, in the determining ratio; therefore P is in the curve; and for the same reason p is a point in the curve.

P R O P. XXXI.

If two right lines QL, Pp , meeting each other in any point L , one of which is parallel to the axis, and the other is inclined to the directrix at any given angle, cut a parabola in the points Q, P and p ; the rectangle under the segment QL and the latus rectum will be to the rectangle under the segments LP, Lp in a constant ratio, wherever the point L be taken.

FIG. 36.

DRAW LX perpendicular to the directrix; and from the center L , at the distance LX , describe a circle. Join QS, XS , and let XS , produced if necessary, meet the circle in T ; and join LT . Draw SO perpendicular to LX ; take OI equal to OX ,

OX , and join SI , which will be equal to SX . Then, LT being equal to LX , and QS equal to QX , LT is to LX as QS is to QX ; therefore LT is parallel to QS ; and because the angle $Q SX$ is equal to the angle SXQ , which is equal to the angle SIX , the triangles QXS , SXI are similar, and IX is to XS , as XS to SQ , or XQ , or as ST to QL ; therefore the rectangle under IX , QL is equal to the rectangle XST , or to the rectangle Dsd ; which, by the preceding proposition, is to the rectangle PLp in a constant ratio; but IX being equal to twice OX the distance of the focus from the directrix, it is equal to the latus rectum; therefore the rectangle under QL and the latus rectum is to the rectangle PLp in a constant ratio.

COR. 1. Hence the rectangle under QL and any other constant quantity is to the rectangle PLp in a constant ratio.

FIG. 31. COR. 2. If the line LP becomes a tangent to the parabola; the square of LP will be to the rectangle under the latus rectum and the segment of the diameter intercepted between the point L and the vertex in the same constant ratio.

L E M M A II.

FIG. 37. If a straight line AB be divided in two points C and D in such a manner, that the rectangle CAD shall be equal to the rectangle DBC , or the rectangle ACB equal to the rectangle BDA , the part AC will be equal to the part BD .

FIRST,

FIG. XXVII.

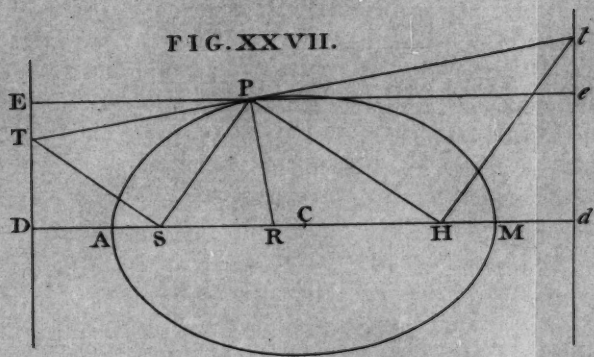


FIG. XXX.

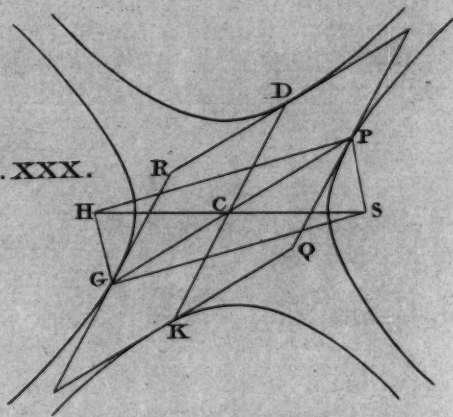


FIG. XXXI.

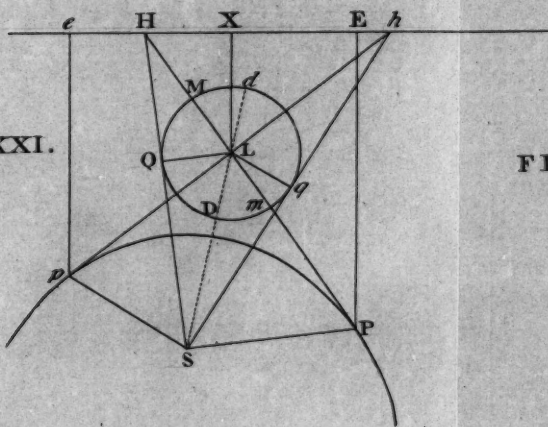


FIG. XXVIII.

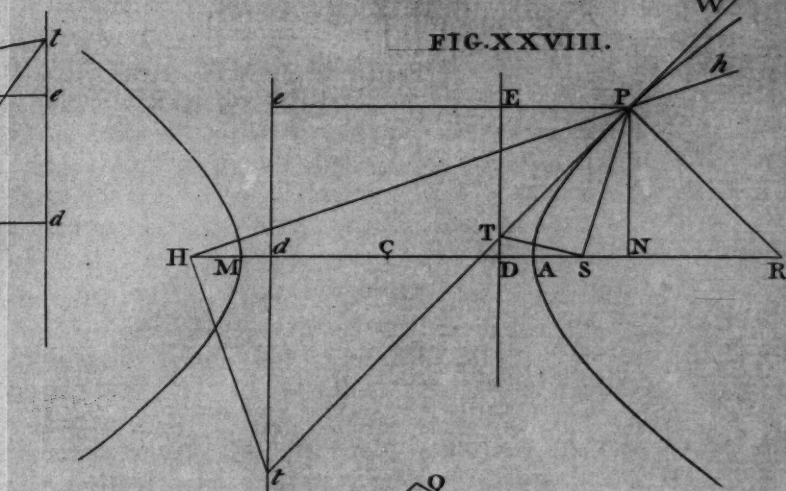


FIG. XXIX.

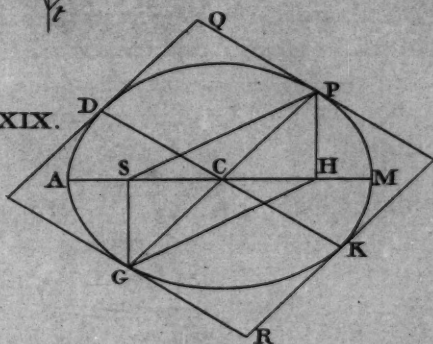
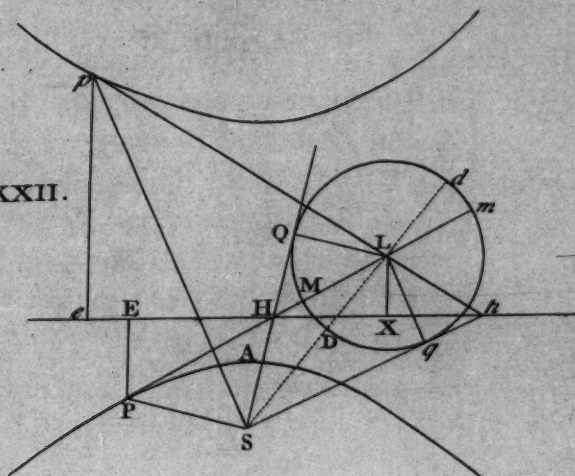


FIG. XXXII.



FIRST, let the rectangle CAD be equal to the rectangle DBC . Bisect CD in E ; and the rectangle CAD , together with the square of EC is equal to the square of AE , and the rectangle DBC , together with the square of ED , is equal to the square of BE ; but the square of ED is equal to the square of EC ; therefore the square of BE is equal to the square of AE , and BE is equal to AE ; and therefore the part BD is equal to the part AC .

Secondly, let the rectangle ACB be equal to the rectangle BDA . Bisect the line AB in E ; and the rectangle ACB , together with the square of EC , is equal to the square of AE ; and the rectangle BDA , together with the square ED , is equal to the square of BE ; but the square of BE is equal to the square of AE ; therefore the square of ED is equal to the square of EC , and ED is equal to EC ; and therefore BD is equal to AC .

P R O P. XXXII.

All right lines drawn parallel to any diameter of the ellipse or hyperbola, which are terminated both ways by the ellipse or opposite hyperbolas, are bisected by the conjugate diameter.

LET ACB be any diameter of the ellipse or hyperbola. Through the vertices A and B draw the tangents AL , BM ; and through the center C draw the diameter DCK parallel to AL , or BM , which will be the conjugate diameter. Through any point N , in the diameter DCK , draw LN parallel to AB , meeting the ellipse or the opposite hyper-

FIG. 38,
39.

hyperbolas in the points P, Q , and the tangents AL, BM in L and M . Then, AL being parallel to CN and BM , and LMN parallel to ACB , AL will be equal to BM , and LN equal NM ; and, Cor. 1. Prop. 30. the square of LA is to the rectangle PLQ as the square of MB is to the rectangle QMP ; but the square of LA is equal to the square of MB ; therefore the rectangle PLQ is equal to the rectangle QMP ; and therefore, Lem. 2. PL is equal to QM ; and if these be taken from the equal lines LN, MN , Fig. 38. and added to them, Fig. 39. PN will be equal to NQ .

COR. 1. If the diameter DK bisect all lines drawn parallel to AB , it will be the conjugate diameter to AB .

FIG. 38. COR. 2. If a right line RDT be drawn through D the vertex of the conjugate diameter parallel to AB , it will touch the ellipse in the point D . For if not, let it meet the curve in some other point d , and Rd will be equal to TD ; but TD is equal to RD ; therefore Rd is equal to RD , which is absurd.

COR. 3. Hence, if the diameter DK be conjugate to any diameter AB , AB will also be conjugate to DK .

P R O P. XXXIII.

Every diameter of a conic section bisects all its ordinates.

FIRST, if the conic section be an ellipse, it is evident from the preceding proposition: for the ordinates of any diameter are parallel to the conjugate diameter.

Secondly,

Secondly, if the section be an hyperbola, of which *ACB* is any diameter; in the tangent *LAR* take *AR* equal to *AL*. Through *L* and *R* draw the lines *PLQ*, *FRG* parallel to *AB*, meeting the opposite hyperbolas in *P*, *Q* and *F*, *G*, and the tangent at the vertex *B* in *M* and *T*. Join *PF*, cutting the diameter in *V*. Then *PF* will be an ordinate which is bisected in *V*: for *PL* is equal to *MQ*, by the preceding proposition, and *FR* equal to *TG*; and the rectangle *FRG* is to the square of *RA* as the rectangle *PLQ* is to the square of *LA*, Cor. 1. Prop. 30. but the square of *RA* is equal to the square of *LA*; therefore the rectangle *FRG* is equal to the rectangle *PLQ*, that is, the rectangle *RFT* is equal to the rectangle *LPM*; therefore Lem. 2. *RF* is equal to *PL*, and *PLRF* is a parallelogram; and therefore *PF* is parallel to the tangent *LAR*, and *PV* is equal to *VF*.

FIG. 39.

Lastly, let the section be a parabola, of which *AN* is any diameter, and *PNQ* an ordinate. Through the vertex *A* draw the tangent *LAM*; and draw *PL*, *QM* parallel to *NA*. Then *PLMQ* is a parallelogram, and *QM* is equal to *PL*; but the rectangle under *LP* and the latus rectum is to the square of *LA* as the rectangle under *MQ* and the latus rectum is to the square of *MA*, Cor. 2. Prop. 31. and the two rectangles being equal, the square of *MA* is equal to the square of *LA*, and *MA* is equal to *LA*; and therefore *QN* is equal to *PN*.

FIG. 40.

COR. 1. If a right line *PQ*, which is terminated by a conic section in the points *P*, *Q*, and which does not pass through the center of the ellipse, be bisected by any diameter, it is parallel to the tangent at the vertex of that diameter; for if it be not parallel to this tangent, let it be parallel to the tangent

FIG. 41.

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gent at the vertex of some other diameter; then PQ would also be bisected by this diameter, which is absurd.

COR. 2. Two right lines terminated by a conic section, which do not pass through the center of the ellipse, cannot bisect each other: for if it be possible, let the two lines DB, RK bisect each other in C , and let CA be the diameter which passes through C ; then both the lines will be parallel to the tangent at the vertex A ; and therefore they are parallel to each other, which is absurd.

COR. 3. A right line bisecting two parallel right lines, which are terminated by a conic section, is a diameter: for a diameter which bisects one of them will bisect the other.

COR. 4. Hence, if a segment of a conic section be given, the diameters and center of the section may be found: for let two parallel right lines be drawn which are terminated by the segment, and the right line which bisects them will be a diameter; in the same manner any other diameter may be found; and if it be parallel to the former, the section will be a parabola; but if the diameters cut each other, the point of intersection will be the center of the ellipse or hyperbola.

COR. 5. If a right line CN , which bisects the two parallels DB, PQ , be produced till it meet the curve in A , and through the point A the right line IAG be drawn parallel to PQ , it will touch the conic section in A .

COR. 6. Hence we have a method of drawing a tangent to a conic section, which shall be parallel to any line which cuts the section in two points. Let PQ be any line cutting the section in two points P, Q ; draw any other line DB parallel to PQ ; bisect PQ, DB in N and C ; join CN , and pro-

produce it till it meet the conic section in A , and through the point A draw IAG parallel to PQ , which will be the tangent required.

P R O P. XXXIV.

If two tangents be drawn at the extremities of any right line which is terminated by a conic section, and which does not pass through the center of the ellipse, they will meet each other in the diameter which bisects that right line.

LET PQ be the right line which is terminated by a conic section in P and Q . Bisect PQ in N , and through N draw the diameter CNT . Through the point P draw the tangent PT meeting the diameter in T , and join TQ , which will touch the section in Q . For draw any other line DCB parallel to PNQ , meeting the lines TP, TQ in L and M . Because the triangles TNP, TCL are similar, as also the triangles TNQ, TCM , TN is to NP as TC is to CL ; and alternately, TN is to TC as NP is to CL ; but TN is to TC as NQ is to CM ; therefore NP is to CL as NQ is to CM ; and alternately, NP is to NQ as CL is to CM ; therefore CM is equal to CL , which is greater than CD , or CB ; and therefore the point M is without the conic section, and the line TQ meets the curve only in one point Q .

FIG. 41.

COR. 1. The tangent IAG at the vertex of the diameter CA , which is terminated by the two tangents TP, TQ , is bisected in A .

COR. 2. If two right lines which touch a conic section

G

section

section meet each other; a right line drawn from their point of concurrence bisecting the line which joins the points of contact will be a diameter of the section.

P R O P. XXXV.

If a right line cutting the hyperbola, or the opposite hyperbolas, meets the asymptotes in two points; the segments between the hyperbola or hyperbolas and asymptotes will be equal.

FIG. 42. **L**ET the line PQ cut the hyperbola, or the opposite hyperbolas in the points P, Q , and meet the asymptotes in R, T ; the segments PR, QT will be equal. If PR be not equal to QT , let one of them, as QT , be the greater; and cut off the part QO equal to PR ; join CO , which being produced will meet the hyperbola in some point q , Cor. Prop. 20. Through the point q draw qpr parallel to QP meeting the curve in p and the asymptote in r . Bisect PQ in N , and draw the diameter CNn , and PQ, pq will be ordinates of that diameter. Then, NQ being equal to NP , and QO equal to PR , NO will be equal to NR ; and QN is to qn as CN to Cn , as NR is to nr ; therefore nq is equal to nr ; but nq is equal to np ; therefore np is equal to nr , which is absurd; and therefore QT is not greater than PR .

COR. 1. If the line TNR be supposed to move from N to A , the points P, Q will coincide in A , and IA will be equal to AG ; therefore when a line touches an hyperbola, the segments between the point of contact and the asymptotes are equal.

COR.

FIG. XXXIII.

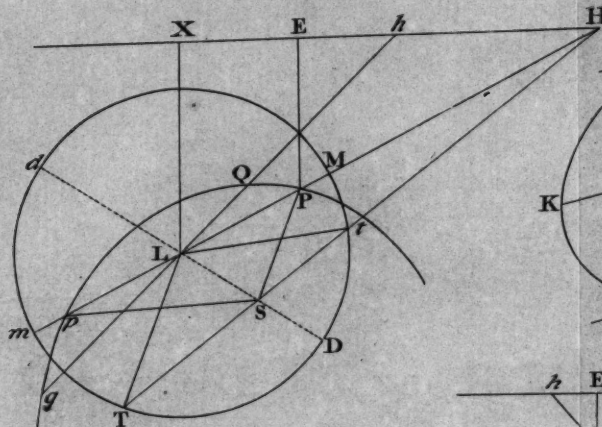


FIG. XXXVIII.

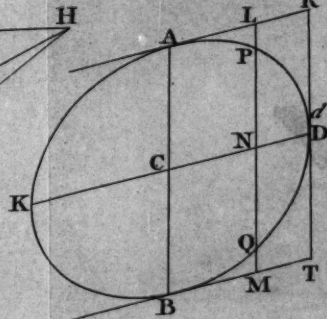


FIG. XXXVI.

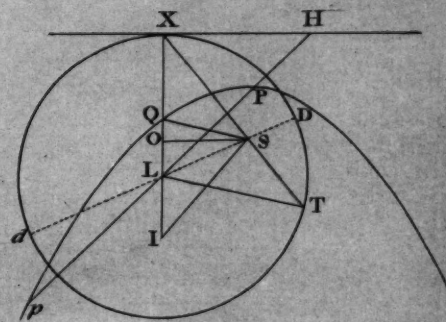


FIG. XXXIV.

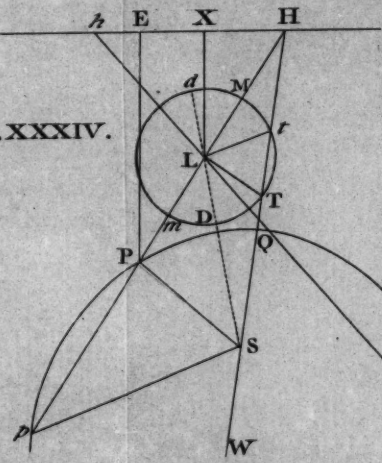


FIG. XXXV.

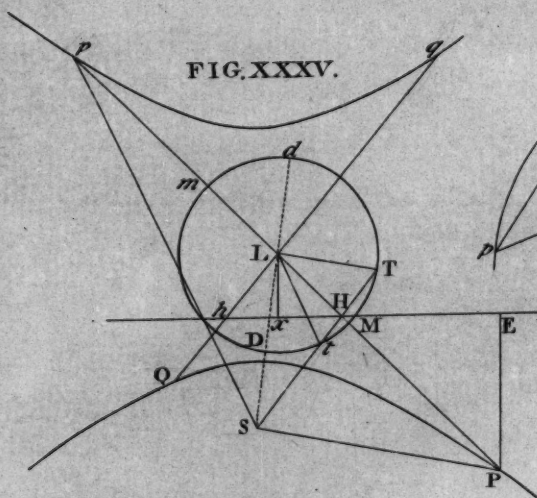


FIG. XXXIX.

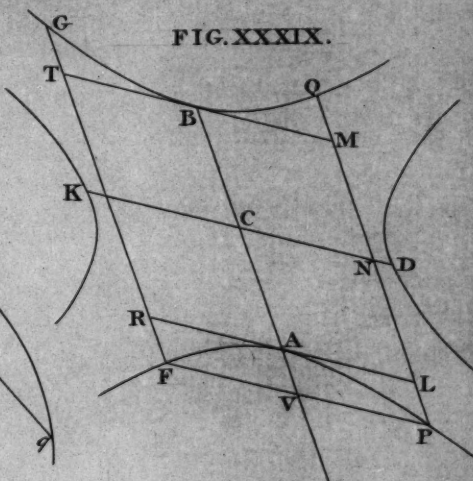


FIG. XXXVII.

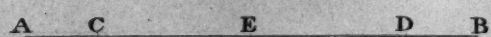


FIG. XL.

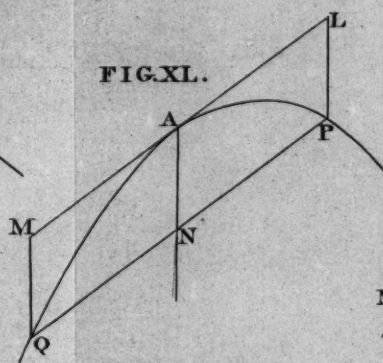
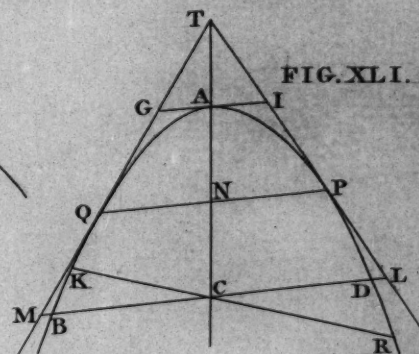


FIG. XLI.



COR. 2. If a right line IAG , meeting the asymptotes in I, G and the hyperbola in A , be bisected in the point A , it will touch the hyperbola; if not, let it meet the curve in some other point a ; and IA will be equal to Ga , and GA equal to Ga , which is absurd.

COR. 3. Because RP is equal to QT , RQ is equal to TP ; therefore the four rectangles PRQ , RPT , QTP , TQR are all equal.

P R O P. XXXVI.

If two right lines be drawn from any point in an hyperbola to the asymptotes, and from any other point, in the same or opposite hyperbola, two other right lines be drawn to the asymptotes parallel to the two former; the rectangle under the first two lines will be equal to the rectangle under the other two.

FROM any point P in the hyperbola PQ draw the lines PL, PH to the asymptotes; and from any other point Q , in the same or in the opposite curve, draw QE, QF parallel to PL, PH . The rectangle under QE, QF will be equal to the rectangle under PL, PH . Join PQ , and let it meet the asymptotes in R and T : and because the triangles TQF, TPH are similar, as also the triangles RPL, RQE, QF is to PH as TQ is to TP , or as RP is to RQ , that is, as PL to QE ; therefore the rectangle under QF, QE is equal to the rectangle under PH, PL .

FIG. 43.

COR. 1. Hence, if from two points P, Q , in the same

same or in the opposite hyperbolas, two right lines PL , QE be drawn to the same or to different asymptotes parallel to the other asymptote; the rectangle CLP will be equal to the rectangle CEQ ; for if the parallelograms $CLPH$, $CEQF$ be completed, the rectangles HPL , FQE , that is, the rectangles CLP , CEQ will be equal.

COR. 2. Because the rectangles are equal, CL is to CE as EQ is to LP , but the parallelograms CEQ , CP are equiangular, therefore they are equal.

COR. 3. Hence, if CP , CQ be joined, the triangles CQF , CQE , CPH and CPL will be equal.

P R O P. XXXVII.

All right lines drawn parallel to an asymptote, which are terminated by two conjugate hyperbolas, are bisected by the other asymptote.

PL. VIII.
FIG. 64.

FROM any point P in the hyperbola AP draw PD parallel to the asymptote HC , and let it meet the conjugate hyperbola BD in D . PD is bisected by the other asymptote CV in I . Join CP , CD . Let ACM , BCb be the two axes; and join AB cutting the asymptote in T . Through A draw the tangent IAa meeting the asymptotes in I and a . Then Aa being equal and parallel to BC , Prop. 21. AB is equal and parallel to aC , and IA is to Ia as AT is to aC , or AB , but IA is half of Ia , therefore AT is half of AB ; and, Cor. 1. preceding proposition, the rectangle CIP is equal to the rectangle CTA , or to the rectangle CTB , which is equal to the rectangle CID ; therefore IP is equal to ID .

P R O P.

P R O P. XXXVIII.

The tangent at the vertex of any diameter of an hyperbola, which is terminated by the asymptotes, is equal to the conjugate diameter.

LET PCG be any diameter of the hyperbola; PL. VIII.
FIG. 64.
from P draw PD parallel to the asymptote HC , meeting the conjugate hyperbola in D ; and join CD . Through P draw the tangent VPH meeting the asymptotes in V and H . VP is equal to PH , Cor. 1. Prop. 35. and VP is to PH as PI is to HC ; therefore PI is half of HC , but it is also half of PD , by the preceding proposition, therefore PD is equal to HC , and DC is equal and parallel to PH , and twice DC , or DK , is equal to VH .

COR. 1. If DK be a conjugate diameter to PG , PG is also conjugate to DK . Join VD , and produce it till it meet the asymptote in h . Then DC being equal and parallel to PH , or VP , VD is equal and parallel to PC , which is equal to Dh , it being the opposite side of the parallelogram $PDhC$; therefore VDH touches the conjugate hyperbola in D , Cor. 2. Prop. 35. and PCG is a conjugate diameter to DCK .

COR. 2. Hence, if two tangents PV , DV be drawn through the vertices of any two conjugate diameters, they will meet in the asymptote; and the asymptotes are diagonals of the parallelogram which is formed by the four tangents.

COR. 3. If the hyperbolas be equilateral, the conjugate diameters will be equal: for the angle HCI will be a right angle, Cor. 2. Prop. 22. it will there-

therefore be in a semicircle, of which VH is the diameter and P the center; therefore CP is equal to PH , which is equal to CD .

P R O P. XXXIX.

If through any point in an asymptote a right line be drawn cutting an hyperbola or opposite hyperbolas; the rectangle under the segments, between the asymptote and the hyperbola or hyperbolas, will be equal to the square of the semidiameter which is parallel to that right line.

FIG. 44. **T**HROUGH any point R in the asymptote draw the right line RT , cutting the hyperbola or opposite hyperbolas in the points P, Q , and the other asymptote in T . The rectangle PRQ will be equal to the square of the semidiameter which is parallel to RT . Let CD be the semidiameter parallel to the line which cuts the hyperbola PAQ , and AC the semidiameter parallel to the line which cuts the opposite hyperbolas. Take any point r in the asymptote, and through r draw rt parallel to RT , cutting the curve or the opposite curves in p, q , and the other asymptote in t . From the points P, p draw PH, PL and pF, pE parallel to the asymptotes; and because the triangles PLR, pEr are similar, as also the triangles PTH, ptF ,
 PR is to PL as pr is to pE , and
 PT is to PH as pt is to pF ; therefore,

Lem. 1. the rectangle RPT is to the rectangle LPH as the rectangle rpt is to the rectangle EpF ; but the rectangle LPH is equal to the rectangle EpF ,

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EpF, Prop. 36. therefore the rectangle *RPT* is equal to the rectangle *rpt*, or the rectangle *PRQ* equal to the rectangle *prq*, Cor. 3. Prop. 35. and if *IAG* be the tangent which is parallel to *RT*, when *P* is taken at *A* the rectangle *PRQ* becomes equal to the square of *IA*, which is equal to the square of *DC*; and when *P* in the opposite hyperbola is at *M*, the rectangle *RPQ* becomes equal to the square of *AC*.

COR. 1. The four rectangles *PRQ*, *RPT*, *QTP*, *TQR* being all equal, are each of them equal to the square of the semidiameter which is parallel to the line *RT*.

COR. 2. As the point *P* recedes from the vertex *A* the line *RP* perpetually decreases, and will become less than any assignable quantity: for the rectangle *PRQ* is equal to a given square, and *PQ* increases without limit; therefore *RP* must decrease without limit. FIG. 42.

P R O P. XL.

If two right lines meeting each other cut or touch a conic section, or opposite sections; the rectangles under the segments between the point of concurrence and the points of intersection, or the squares of the tangents will be to each other as the squares of the semidiameters to which the lines are parallel.

IF the lines be parallel to any of the diameters of the ellipse, or to any of the diameters of the opposite hyperbolas; the proposition is evident from prop.

Prop. 30. because the lines which meet each other make the same angles with the directrix as those which pass through the center, and the latter are bisected in the center. But if either of the lines PLQ , LRT , or both the lines PLQ , NLM be parallel to some of the conjugate diameters of the hyperbola; produce QLP till it meet the asymptote in G , and through G draw FGH parallel to LRT , meeting the opposite curves in F and H . Let CB , CD and CA be the semidiameters which are parallel to QP , MN and RT . Then, Prop. 30. the rectangle PLQ is to the rectangle RLT as the rectangle PGQ is to the rectangle FGH , or as the square of CB to the square of CA , by the preceding proposition. In the same manner it may be proved, that the rectangle RLT is to the rectangle NLM as the square of CA to the square of CD ; therefore the rectangle PLQ is to the rectangle NLM as the square of CB is to the square of CD .

If the lines touch the conic section or opposite sections, the squares of the tangents will be to each other as the rectangles under the segments of any two lines drawn parallel to them, which meet each other, and cut the section or opposite sections; and therefore they are as the squares of the semidiameters to which they are parallel.

FIG. 45. COR. If two right lines IQ , IN , meeting each other in I , touch an ellipse or hyperbola in Q , N , and are parallel to two other lines VT , VN which meet each other in V , and touch the ellipse or opposite hyperbola or hyperbolas in T , N ; IQ will be to IN as VT is to VN : for the squares of IQ , IN are to each other as the semidiameters to which they are parallel, and the squares of VT , VN are in the same ratio,

PROP.

P R O P. XLI.

If an ordinate be drawn to any diameter of an Ellipse or an Hyperbola; the rectangle under the abscissæ will be to the square of the semi-ordinate as the square of the diameter is to the square of its conjugate.

LET ACM be any diameter of an ellipse or hyperbola, to which PNQ is an ordinate; and let DCK be the conjugate diameter, which is parallel to PNQ . Then, by the preceding proposition, the rectangle ANM is to the rectangle PNQ , or the square of PN , as the square of CA to the square of CD , or as the square of AM to the square of DK . FIG. 47;
48.

COR. 1. Because the parameter is a third proportional to the diameter and its conjugate, the rectangle under the abscissæ is to the square of the semi-ordinate as the diameter is to the parameter.

COR. 2. The two conjugate diameters being constant, the rectangle under the abscissæ will vary as the square of the ordinate.

COR. 3. Those ordinates which are at equal distances from the center are equal; and those which are nearer to the center are greater in the ellipse, and less in the hyperbola than those which are more remote.

COR. 4. If the hyperbola be equilateral, the rectangle under the abscissæ will be equal to the square of the semi-ordinate.

P R O P. XLII.

If an ordinate be drawn to any diameter of the Parabola; the square of the semi-ordinate will be equal to the rectangle under the abscissa and the parameter.

FIG. 49. **L**ET AN be any diameter of the parabola, to which PNQ is an ordinate. Draw the parameter TSV , cutting the diameter in F ; join SA , and let the diameter be produced till it meet the directrix in D . Because TF is half of TV , Prop. 33. and AF is half of DF , or half of TF , Prop. 26. AF is to TF as TF is to TV , and the rectangle under AF and TV is equal to the square of TF ; but the rectangle under AF and TV is to the rectangle under AN and TV as the square of TF is to the square of PN , Cor. 1. Prop. 31. therefore the rectangle under AN and TV is equal to the square of PN .

COR. 1. Because TV is equal to twice TF , or to four times SA , the rectangle under AN and four times SA is equal to the square PN .

COR. 2. If the tangent at the vertex of any diameter meets another diameter produced, the square of the tangent will be equal to the rectangle under that part of the diameter which is produced and the parameter which belongs to the first diameter: for NI is a parallelogram, and AI, IQ are equal to QN, AN ; therefore the square of AI is equal to the rectangle under IQ and TV .

COR. 3. The parameter being constant, the square of the ordinate varies as the abscissa.

P R O P.

P R O P. XLIII.

If AR be taken in the tangent at the vertex FIG. 47.
48.
49.
of any diameter of a conic section equal to
the parameter of that diameter, and a line
be drawn from R to the other vertex in the
ellipse and hyperbola, which line is parallel
to the diameter in the parabola; and if the
ordinate QNP meets the line RM in L ;
the rectangle under the abscissa AN and
the line NL will be equal to the square of
the semi-ordinate PN .

IF the conic section be a parabola, the propo-
sition is manifest. If the section be an ellipse or
an hyperbola, the rectangle ANM is to the square
of PN as AM is to AR , as NM to NL , or as the
rectangle ANM to the rectangle AN, NL ; there-
fore the square of PN is equal to the rectangle AN ,
 NL .

COR. If the diameter of an ellipse or an hyper-
bola becomes infinite, and the abscissa AN be fi-
nite, RL will be parallel to AN ; therefore NL will
be equal to AR , and the ordinate PNQ will be
equal to the ordinate of a parabola, of which AN
is the diameter and AR the parameter of that dia-
meter. Hence the ellipse or hyperbola will have
the same properties as the parabola at all finite dis-
tances from the vertex.

PROP. XLIV. PROB. VI.

Two right lines being given, one of which is bisected by the other; to describe a parabola, of which the right line bisected shall be an ordinate, and the other line the diameter.

FIG. 49. **L**ET AN , PQ be the two given lines one of which is bisected in N . Through A draw IAR parallel to QNP , and take AR a third proportional to AN and PN ; produce NA to D , so that AD may be a fourth part of AR . Through the point D draw DX perpendicular to DN . Let the angle DAI be less than the angle DAR ; make the angle IAS equal to the angle IAD , and take AS equal to AD . Then if a parabola be described, of which S is the focus, and DX the directrix; AN will be a diameter of that parabola, and PNQ an ordinate: for SA being equal to AD , A is a point in the parabola, Cor. Prop. 17. and because the angle SAI is equal to the angle IAD , AI is a tangent to the parabola in the point A , Cor. 1. Prop. 25. therefore QNP is parallel to the ordinates of the diameter which passes through A ; and PN being a mean proportional between AN and four times AS , it must be equal to the semi-ordinate, Cor. 1. Prop. 42.

COR. 1. Hence, if the parameter of any diameter be given, and the angle which that diameter makes with its ordinates, the parabola may be described: for any abscissa being taken in the diameter, the semi-ordinate may be found, by taking a mean

mean proportional between the abscissa and the given parameter.

COR. 2. If in any figure PAQ all the right lines PQ, TV &c. which are inclined to the indefinite right line AN at a given angle, are bisected by the line AN , and the squares of PN, TF , &c. are as the segments AN, AF , &c. the curve PAQ which passes through the extremities of these lines is a parabola, of which the indefinite right line AN is a diameter, and the lines PQ, TV , &c. are ordinates: for describe the parabola having one of these lines PQ for an ordinate, and AN an abscissa. Then, because the square of PN is to the square of TF as AN is to AF , TFV is also an ordinate of that parabola. In like manner it may be proved, that all the lines PQ, TV , &c. are terminated by the parabola.

COR. 3. From this proposition we have a method of finding two mean proportionals between two given right lines. Let P, Q be the two given lines. Describe two parabolas BAC, DAC , the axes of which AE, AF are perpendicular to each other, and of which P, Q are the parameters. From the point C , in which the two curves intersect each other, draw CE, CF perpendicular to the axes, and they will be the proportionals required: for P is to CE , or AF , as AF is to AE , or CF , and AF is to CF as CF is to Q .

FIG. 50.

PROP.

P R O P. XLV.

If a right line be drawn from the focus of a parabola perpendicular to any tangent; it will be a mean proportional between the distance of the point of contact from the focus, and the distance of the focus from the vertex.

FIG. 51. **L**ET PT touch the parabola in any point P , and let it meet the axis in T . Draw PE perpendicular to the directrix; join SE cutting the tangent in the point Y , and join AT . Then ST will be perpendicular to PT : for, SP being equal to PE , and the angle SPY equal to the angle EPY , the triangles SPY , EPY are equal; therefore SY is equal to ET , and the angle STP is equal to the angle PTE ; therefore they are each of them a right angle. And SA being equal to AD , AT is parallel to DE , and SAT is a right angle; therefore the triangles SAT , STT are similar, and ST , or SP , is to SY as SY is to SA .

COR. 1. Because the rectangle under SP , SA is equal to the square of SY , and SA is constant in the same parabola, the squares of the perpendiculars from the focus upon the tangents will be as the distances of the points of contact from the focus.

COR. 2. If PR be drawn from the point of contact perpendicular to the tangent, meeting the axis in R ; SR will be equal to ST , or SP : for RP is parallel to SE , and $RPES$ is a parallelogram; therefore SR is equal to PE , or SP .

COR.

FIG. XLVI.

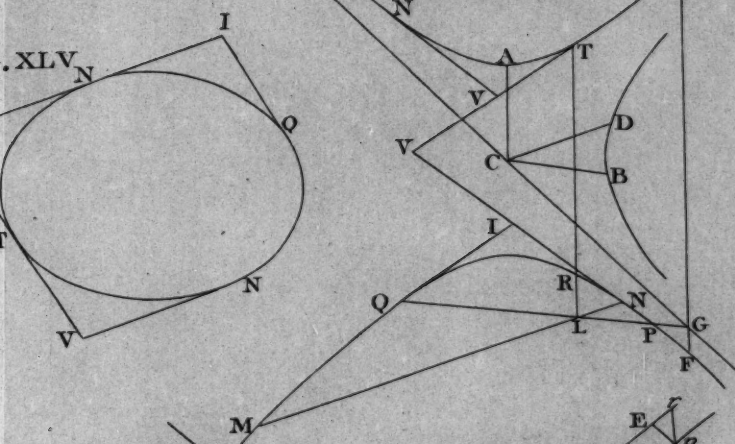


FIG. XLIV.

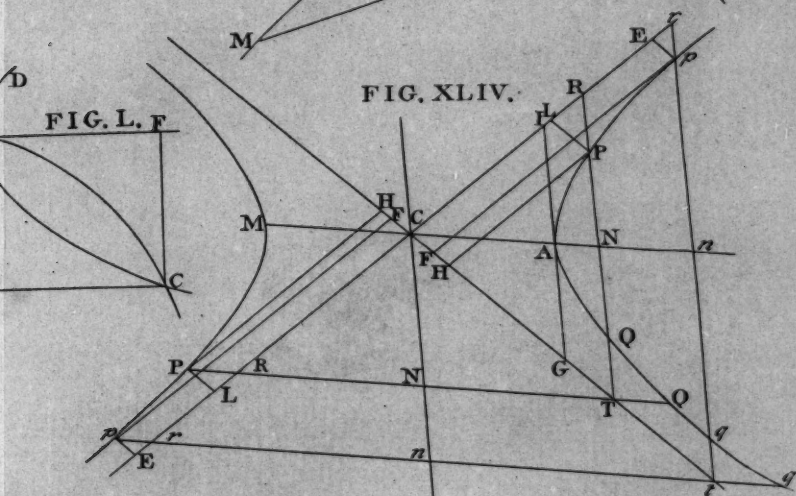
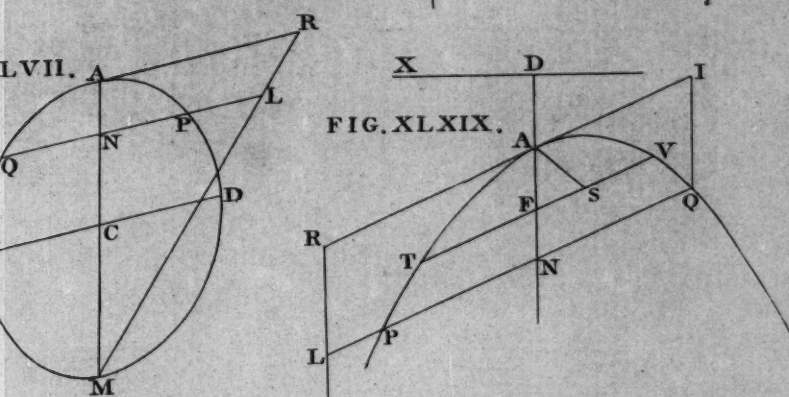


FIG. XLXIX.



COR. 3. The normal PR is equal to twice the perpendicular ST .

COR. 4. The subnormal RN is equal to half the latus rectum: for SR is equal to PE , or ND ; and from each be taken the common part SN , NR will be equal to SD , or to half the latus rectum.

P R O P. XLVI.

If a tangent to any point in the parabola meets a diameter, and an ordinate be drawn to that diameter from the point of contact; the segment of the diameter between the vertex and the tangent will be equal to the abscissa.

LET TP , which touches the parabola in any point P , meet the diameter NA in T ; and draw the ordinate PN . NA will be equal to AT . FIG. 52.

Through the vertex A draw the tangent AI meeting PT in I ; join AP , and draw the diameter IG , cutting the line AP in F , and the ordinate PN in G . Because IG is a diameter which passes through the point of concurrence of the two tangents AI , AI , it will bisect AP in F , Cor. 2. Prop. 34. and the triangles FIA , FGP being equiangular, FI will be equal to PG ; but AI is equal to NG ; therefore PG , GN are equal; and AI is half of NP ; but TA is to TN as AI is to NP ; therefore TA is half of TN , or TA is equal to AN .

COR. Hence we have another method of drawing a tangent to the parabola from a given point without. Let T be the given point; draw TAN parallel to the axis, cutting the parabola in A :
through

through A draw the tangent AI ; take AN equal to AT ; and through N draw NP parallel to AI , meeting the curve in P ; and join TP , which will be the tangent required.

P R O P. XLVII.

If two right lines be drawn from the foci of an ellipse or an hyperbola perpendicular to any tangent; they will meet the tangent in the circumference of a circle, which has the transverse axis for a diameter.

FIG. 53. **54.** **L**ET PT touch the ellipse or hyperbola in any point P . Join SP, PH ; and draw ST, HZ perpendicular to the tangent: produce ST till it meet HP in the point W ; and join CT . Because the angles SPY, YPW are equal, Prop. 27. and PT is common to the two triangles SPY, YPW , PW is equal to PS , and ST equal to YW ; therefore HW is equal to the transverse axis, Prop. 14. and because SC is equal to CH , and ST equal to YW , CT is parallel to HW ; and SC is to CT as SH is to HW ; but SC is half of SH ; therefore CT is half of HW , or AM . And if from the center C , at the distance CA , a circle be described, it will pass through the point T . In the same manner it may be proved that it will pass through Z .

COR. If the diameter DCK be parallel to the tangent at P ; it will cut off from SP, PH the segments PE, PI , each of them equal to half the transverse axis: for $CEPZ$ and $CYPI$ are parallelograms; therefore PE, PI are equal to CZ, CT , each of which is equal to CA .

P R O P.

P R O P. XLVIII.

The rectangle under the perpendiculars, which are drawn from the foci of an ellipse or an hyperbola to any tangent, is equal to the square of half the conjugate axis.

THE same construction remaining as in the last proposition, produce ZC till it meet YS produced in G ; and let CB be the conjugate semi-axis. Because GS is parallel to HZ , the triangles CSG, CHZ are equiangular; and CS being equal to CH , SG is equal to HZ , and CG is equal to CZ , or CA ; therefore the point G is in the circumference of the circle; and the rectangle GST is equal to the rectangle ASM , that is, the rectangle under HZ, ST is equal to the square of BC .

FIG. 53.
54.

COR. 1. The square of ST is to the rectangle ST, HZ , or the square of BC , as ST is to HZ , or, because the triangles SPY, HPZ are similar, as SP to HP ; therefore the square of ST is equal to

$$BC^2 \times \frac{SP}{HP}$$

COR. 2. Hence, the square of BC being constant, the square of the perpendicular ST will vary as SP directly, and as HP inversely.

P R O P. XLIX.

If a tangent to any point in an ellipse or an hyperbola meets a diameter, and from the point of contact an ordinate be drawn to that diameter; the semidiameter will be a mean proportional between the segments of the diameter, which are intercepted between the center and the ordinate, and between the center and the tangent.

FIG. 55.
56. **L**ET the right line PT touch the ellipse or hyperbola in any point P , and let it meet the diameter MA in T ; and from the point P draw PN an ordinate to the diameter MA . CN is to CA as CA is to CT . Through the vertices A, M draw the tangents AI, ML , meeting the tangent PT in I and L ; and take CO on the opposite side of the center equal to CN . Then, Cor. Prop. 40. IP is to IA as LP is to LM ; and alternately, IP is to LP as IA is to LM ; and because the lines AI, NP, ML are parallel, AN is to NM as TA is to TM ; and by division, Fig. 55. and by composition, Fig. 56, ON is to AN as AM is to TA ; and by taking the halves of the antecedents, CN is to AN as CA is to TA ; and by composition, Fig. 55. and by division, Fig. 56. CA is to CN as CT is to CA ; and by inversion, CN is to CA as CA is to CT .

COR. I. The segment of the diameter intercepted between the ordinate and the center, the two abscissæ, and the segment between the ordinate and the tangent are proportionals. CN is to AN as NM is to TN ; for CN is to CM as CM is to CT ; and

and by composition, CN is to NM as CM is to TM ; and alternately, CN is to CM as NM is to TM ; and by division, CN is to AN as NM is to TN .

COR. 2. The segments TA , TN , TC and TM are proportionals: for TC is to AC as AC is to NC ; and by division, TA is to TC as AN is to AC ; and alternately, TA is to AN as TC is to AC ; and by composition, TN is to TA as TM is to TC ; and by inversion, TA is to TN as TC is to TM .

COR. 3. From this proposition we have another method of drawing a tangent to an ellipse or an hyperbola from a given point without. Let T be the given point; draw the diameter CT cutting the curve in A . Through the point A draw the tangent AI ; take CN a third proportional to CT , CA ; and draw NP parallel to AI , meeting the curve in P ; and join TP , which will touch the curve in the point P .

P R O P. L.

If a tangent to any point in the hyperbola meets a conjugate diameter, and an ordinate be drawn from the point of contact to that diameter; the conjugate semidiameter will be a mean proportional between the segments, which are intercepted between the ordinate and the center, and between the center and the tangent.

LET the tangent PT , which meets the diameter AM in T , be produced till it meet the conjugate diameter Bb in t . Draw the ordinate

FIG. 56.

Pn parallel to AM . Because CN is to CA as CA is to CT , by the preceding proposition, the square of CN is to the square of CA as CN is to CT ; and by division, the difference of the squares of CN and CA is to the square of CA as TN is to CT , or, because PN, Ct are parallel, as PN is to Ct , or as Cn to Ct ; but the square of Cn is to the square of CB as the rectangle ANM is to the square of CA , Prop. 41. or as the difference of the squares of CN and CA is to the square of CA ; therefore the square of Cn is to the square of CB as Cn is to Ct ; and therefore the three lines Cn, CB and Ct are proportionals, that is, Cn to CB as CB to Ct .

P R O P. LI.

FIG. 55. If two tangents AI, ML be drawn through the vertices of any diameter of an ellipse or hyperbola, meeting any other tangent PT in I and L ; the rectangle under the tangents AI, ML will be equal to the square of the semidiameter CB to which they are parallel; and the rectangle under IP, PL , the segments of the tangent which they meet, will be equal to the square of the semidiameter CD parallel to this tangent.

IF the tangent TP be parallel to AM in the ellipse, the proposition is manifest; for each of the tangents AI, ML will be equal to the semidiameter CB . But if TP be not parallel to the diameter AM ; let it meet it in the point T , and the diameter Bb in t . From the point P draw the semi-or-

dinates

dinates PN , Pn to the diameters AM , Bb . Then, Cor. 2. Prop. 49. TA is to TN as TC is to TM ; and because the lines AI , NP , Ct and ML are parallel, AI is to NP , or Cn , as Ct is to ML ; therefore the rectangle under AI , ML is equal to the rectangle under Cn , Ct , or to the square of CB , Prop. 49, and 50. Secondly,

AI is to IP as CB is to CD , Cor. Prop. 40. and ML is to PL as CB is to CD ; therefore the rectangle AI , ML is to the rectangle IP , PL as the square of CB is to the square of CD ; but the rectangle AI , ML is equal to the square of CB ; therefore the rectangle IP , PL is equal to the square of CD .

P R O P. LII.

If a tangent to any point in an ellipse or an hyperbola meets two conjugate diameters; the rectangle under the segments of the tangent, between the point of contact and the diameters, is equal to the square of the semidiameter which is parallel to the tangent.

LET the tangent PT meet the two conjugate diameters AM , Bb in T and t . From the point P draw the ordinate PNQ to the diameter AM ; and through the points A , M draw the tangents AI , ML , meeting the tangent PT in I and L ; and draw the diameter DCK parallel to PT . Because CN is to AN as NM is to TN , Cor. 1. Prop. 49. Pt is to PI as PL is to PT ; therefore the rectangle TPt is equal to the rectangle IPL , which

FIG. 55.
56.

Pn parallel to AM . Because CN is to CA as CA is to CT , by the preceding proposition, the square of CN is to the square of CA as CN is to CT ; and by division, the difference of the squares of CN and CA is to the square of CA as TN is to CT , or, because PN, Ct are parallel, as PN is to Ct , or as Cn to Ct ; but the square of Cn is to the square of CB as the rectangle ANM is to the square of CA , Prop. 41. or as the difference of the squares of CN and CA is to the square of CA ; therefore the square of Cn is to the square of CB as Cn is to Ct ; and therefore the three lines Cn, CB and Ct are proportionals, that is, Cn to CB as CB to Ct .

P R O P. LI.

FIG. 55. If two tangents AI, ML be drawn through
56. the vertices of any diameter of an ellipse or hyperbola, meeting any other tangent PT in I and L ; the rectangle under the tangents AI, ML will be equal to the square of the semidiameter CB to which they are parallel; and the rectangle under IP, PL , the segments of the tangent which they meet, will be equal to the square of the semidiameter CD parallel to this tangent.

IF the tangent TP be parallel to AM in the ellipse, the proposition is manifest; for each of the tangents AI, ML will be equal to the semidiameter CB . But if TP be not parallel to the diameter AM ; let it meet it in the point T , and the diameter Bb in t . From the point P draw the semi-or-
dinates

dinates PN , Pn to the diameters AM , Bb . Then, Cor. 2. Prop. 49. TA is to TN as TC is to TM ; and because the lines AI , NP , Ct and ML are parallel, AI is to NP , or Cn , as Ct is to ML ; therefore the rectangle under AI , ML is equal to the rectangle under Cn , Ct , or to the square of CB , Prop. 49, and 50. Secondly,

AI is to IP as CB is to CD , Cor. Prop. 40. and ML is to PL as CB is to CD ; therefore the rectangle AI , ML is to the rectangle IP , PL as the square of CB is to the square of CD ; but the rectangle AI , ML is equal to the square of CB ; therefore the rectangle IP , PL is equal to the square of CD .

P R O P. LII.

If a tangent to any point in an ellipse or an hyperbola meets two conjugate diameters; the rectangle under the segments of the tangent, between the point of contact and the diameters, is equal to the square of the semidiameter which is parallel to the tangent.

LET the tangent PT meet the two conjugate diameters AM , Bb in T and t . From the point P draw the ordinate PNQ to the diameter AM ; and through the points A , M draw the tangents AI , ML , meeting the tangent PT in I and L ; and draw the diameter DCK parallel to PT . Because CN is to AN as NM is to TN , Cor. 1. Prop. 49. Pt is to PI as PL is to PT ; therefore the rectangle TPt is equal to the rectangle IPL , which

FIG. 55.
56.

which is equal to the square of CD , by the preceding proposition.

COR. Hence, if a right line TP which touches an ellipse or hyperbola meets two diameters AM , Bb in T and t , and the rectangle TPt is equal to the square of the semidiameter which is parallel to the tangent, AM , Bb are conjugate diameters.

PROP. LIIL. PROB. VII.

Two right lines bisecting each other being given; to describe an ellipse or an hyperbola, of which the two given lines shall be conjugate diameters.

FIG. 57, 58. IF the two given lines be at right angles to each other, they will be the axes; and the ellipse or hyperbola may be described by Prop. 15. But if the lines PG , DK , bisecting each other in any point C , be not at right angles to each other; through the point P draw PT parallel to DK . Take PQ in the line CP , towards the center in the hyperbola, and the contrary way in the ellipse, a third proportional to CP , CD . Bisect CQ in the point V ; and draw VR perpendicular to CQ , meeting the line PT in R . From the center R , at the distance RQ , or RC , describe a circle cutting the line PT in T and t . Draw the lines CT , Ct , to which draw PN , Pn perpendicular. In CT take CA a mean proportional between CN and CT ; and in Ct take CB a mean proportional between Cn and Ct . Make CM equal to CA , and Cb equal to CB . Then because the angle TCt is in a semicircle, it is a right angle; and the lines AM , Bb bisect each other; therefore describe an ellipse or an hyperbola,

perbola, of which AM, Bb are the axes; and PG, DK will be conjugate diameters of that ellipse or hyperbola: for PN is to CB as CB is to Ct ; therefore the square of PN is to the square of CB as PN is to Ct , that is, by similar triangles, as NT to CT ; but CN being to CA as CA to CT , CN is to CT as the square of CN is to the square of CA ; and by division, NT is to CT as the difference of the squares of CA and CN is to the square of CA , or as the rectangle ANM to the square of CA ; therefore the square of PN is to the square of CB as the rectangle ANM is to the square of CA ; and consequently PN must be a semi-ordinate, and the point P is in the curve, to which TP is a tangent, Prop. 49. and because KD is parallel to TP , and the rectangle TPt is equal to the rectangle CPQ , or to the square of CD , KD is a conjugate diameter to PG , by the preceding proposition.

COR. 1. Hence, if any two conjugate diameters of an ellipse or an hyperbola be given, the axes may be found.

COR. 2. If the right line PNQ drawn through FIG. 55,
any point in the line AM between the points A 56.
and M , or in MA produced, making any given angle with AM , is bisected by it in N , and the square of PN or QN is to the rectangle ANM in any given ratio; the points P, Q will be in an ellipse in the first case, and in an hyperbola in the second, of which AM is a diameter, and PQ an ordinate: for bisect AM in C ; through C draw BCb parallel to PQ , and take CB, Cb such, that the square of CB , or Cb , may be to the square of CA in the given ratio; and describe an ellipse or an hyperbola, of which AM, Bb may be conjugate diameters. Then because PQ is parallel to Bb , and the square of PN is to the rectangle ANM as the
square

square of CB to the square of CA , PNQ is an ordinate to the diameter AM ; and therefore the points P, Q are in the ellipse or hyperbola, of which AM, Bb are conjugate diameters.

P R O P. LIV.

FIG. 55. If a right line PT touches an ellipse or an
56. hyperbola in any point P , and PR be drawn through the point of contact perpendicular to the tangent, meeting the two axes in R and r ; the rectangle under the normals PR, Pr will be equal to the square of the semidiameter CD , which is parallel to the tangent.

LET the tangent meet the two axes in T and t . Then the triangles TPR, TCt are similar; for they have each of them a right angle, and they have a common angle at T , Fig. 55. and the angles at T are vertical, Fig. 56. and the right angled triangles TCt, rPt , which have a common angle at t , are also similar; and RP is to TP as Pt is to rP ; therefore the rectangle under RP, rP is equal to the rectangle TPt , which is equal to the square of CD , Prop. 52.

P R O P.

P R O P. LV.

If a right line be drawn from the center of an ellipse or hyperbola perpendicular to any tangent; the rectangle under the perpendicular and the normal, which is terminated by either of the axes, is equal to the square of half the other axis.

DRAW CT perpendicular to the tangent; and the rectangle under CT and Pr will be equal to the square of CA ; and the rectangle under CT and PR equal to the square of CB . FIG. 55,
56.

From the point P draw PN, Pn perpendicular to the axes; and because the angle at r is common to the two right angled triangles rnP, rPt , the triangle rnP is similar to the triangle rPt , which is similar to Tct , or to TYC ; therefore CT is to Ct as Pn , or Cn , is to Pr ; and the rectangle under CT, Pr is equal to the rectangle under Cn and Ct , which is equal to the square of CA , Prop. 49. Secondly, because the angle PRN is equal to the angle rPn , which is equal to the angle rtP , the right angled triangles RPN, CtT are similar; and CT is to Ct as PN , or Cn , is to PR ; therefore the rectangle under CT, PR is equal to the rectangle under Ct and Cn , which is equal to the square of CB . Prop. 49, and 50.

COR. 1. The normals are to each other inversely as the squares of the axes by which they are terminated: for, the perpendicular CT being common to the two rectangles, Pr is to PR as the square of CA to the square of CB .

K

COR.

COR. 2. The perpendicular CY varies inversely as the normal, which is terminated by either of the axes; the rectangle under CY, PR and CY, Pr being each of them equal to a given square.

P R O P. LVI.

If a right line be drawn from the center of an ellipse or an hyperbola perpendicular to any tangent; the rectangle under the perpendicular and the femidiameter which is parallel to the tangent is equal to the rectangle under the semi-axes.

FIG. 55, 56. **T**HE same construction remaining as in the preceding propositions, the square of CY is to the rectangle under CY, PR as CY is to PR ; and the rectangle under CY, Pr is to the rectangle under PR, Pr , as CY to PR ; therefore the square of CY is to the rectangle under CY, PR , or the square of CB , as the rectangle under CY, Pr is to the rectangle under PR, Pr or as the square of CA to the square of CD ; therefore CY is to CB as CA to CD ; and the rectangle under CY, CD is equal to the rectangle under CA, CB .

FIG. 59, 60. COR. 1. If a parallelogram be formed by drawing tangents through the vertices of any two conjugate diameters; it will be equal to the rectangle under the axes. Let PG, DK be any two conjugate diameters; and draw the tangents through the vertices, which will be parallel to the conjugate diameters; then the four parallelograms DP, PK, KG, GD will be equal to each other; therefore the parallelogram DP is a fourth part of the parallelogram VW ; and

and the parallelogram DP is equal to the rectangle under PV, CY , or CD, CY , which is equal to the rectangle under AC, CB , a fourth part of the rectangle under the axes.

COR. 2. Hence the parallelograms, which are formed by drawing tangents through the vertices of conjugate diameters, are equal.

COR. 3. If DP, PK, KG, GD be joined, the figure $DPKG$ will be a parallelogram, which is half of the parallelogram VW ; therefore all the parallelograms, which are formed by joining the vertices of conjugate diameters, are equal.

P R O P. LVII.

If two right lines be drawn from any point in an ellipse or an hyperbola to the foci; they will contain a rectangle equal to the square of the semidiameter parallel to the tangent drawn through that point.

FROM any point P in the ellipse or hyperbola draw PS, PH to the foci, and let DK be the diameter which is parallel to the the tangent PT . The rectangle under SP, PH will be equal to the square of CD . Draw the lines SY, HZ from the two foci perpendicular to the tangent; and draw PF perpendicular to the tangent, meeting the diameter DK in F ; and PF will be equal to the line which is drawn from the center perpendicular to the tangent. Then, because the rectangle under AC, CB , or PE, CB , is equal to the rectangle under CD, PF , by the preceding proposition, PE is to PF as

FIG. 53,
54.

CD is to CB , and because the triangles SPT , PEF , HPZ are similar,

SP is to ST as PE to PF , or as CD to CB , and PH is to HZ as PE to PF , or as CD to CB ; therefore the rectangle SP , PH is to the rectangle ST , HZ , or the square of CB , as the square of CD is to the square of CB ; therefore the rectangle under SP , PH is equal to the square of CD .

P R O P. LVIII.

FIG. 51, 53, 54. If a right line RL be drawn from the point R , where the normal meets the axis of the parabola or the transverse axis of the ellipse and hyperbola, perpendicular to the distance SP from the focus to the point of contact; it will cut off from SP the segment PL equal to half the latus rectum.

FIG. 51. **F**IRST, if the section be a parabola, draw PN perpendicular to the axis; and because SR is equal to SP , Cor. 2. Prop. 45. the angle SPR is equal to the angle SRP ; and because the angles PLR , PNR are right angles, and PR is common to the two triangles PLR , PNR , the triangles are equal, and PL is equal to RN , or to half the latus rectum, Cor. 4. Prop. 45.

FIG. 53, 54. Secondly, if the section be an ellipse or an hyperbola, let PR meet the diameter which is parallel to the tangent in the point F . Then the triangles PRL , PEF will be similar; for the angles at L and F are right angles, the angle at P , Fig. 53. is common, and the angles at P , Fig. 54. are vertical; therefore PE , or AC , is to PF as PR is to PL , and the rect-

rectangle under AC , PL is equal to the rectangle under PF , PR , or to the square of BC , Prop. 55. and AC is to BC as BC is to PL ; therefore PL is equal to half the latus rectum, Prop. 7.

COR. Hence, if PL be taken in any of the conic sections equal to half the latus rectum, and two right lines be drawn from the points P and L , one of which is perpendicular to the tangent at the point P , and the other perpendicular to PS , they will meet each other in the axis.

P R O P. LIX. P R O B. VIII.

The distance of any point in a conic section from the focus, the latus rectum, and the position of the tangent at that point being given; to describe the conic section.

IF the given distance be perpendicular to the tangent, it will be in the direction of the axis, and the conic section may be described by Cor. 1. Prop. 4. But if the distance be not perpendicular to the tangent, let SP be the given distance, and PY the tangent at the point P . Take PL equal to half the latus rectum; from the points P and L draw the lines PR , LR perpendicular to PS and PY ; and the point R where they meet each other is in the axis, by the Cor. to the last proposition. Join SR , which produce indefinitely. Make the angle ZPH equal to the angle SPT ; and SR will meet PH in the direction SR , in the opposite direction, or it will be parallel to it: in the first case the conic section will be an ellipse, in the second an hyperbola, and in the third a parabola. If the section be an ellipse or an hyperbola, bisect SH in the point

FIG. 51.
53.
54.

FIG. 53.
54.

point C , and draw CE parallel to the tangent, meeting the line PS in E , and PE will be equal to half the transverse axis; therefore take CA , CM each of them equal to PE ; and AM will be the transverse axis. If the section be a parabola, take SA equal to a fourth part of the latus rectum, and the point A will be the vertex of the axis. The position of the axis, and the distance of the focus from the vertex being found, the conic section may be described by Cor. 1. Prop. 4. or, the transverse axis and the two foci being found in the ellipse and hyperbola, the conic section may be described by Prop. 15, and 16.

P R O P. LX.

If two ellipses or hyperbolas have a common diameter, and an ordinate be drawn to each of the curves through the same point in the common diameter; the ordinates will be to each other as the conjugate diameters.

FIG. 61, 62. **L**ET AP , AQ be two ellipses, or two hyperbolas, having a common diameter AM . Through any point N in that diameter draw NP , NQ ordinates to the two curves; and let CB , CD be the semidiameters which are parallel to NP , NQ . Then NP will be to NQ as CB is to CD : for the square of NP is to the square of CB as the rectangle ANM is to the square of CA , or as the square of NQ is to the square of CD ; therefore NP is to CB as NQ is to CD ; and alternately, NP is to NQ as CB is to CD , or as $2CB$ to $2CD$.

COR.

FIG. 11.

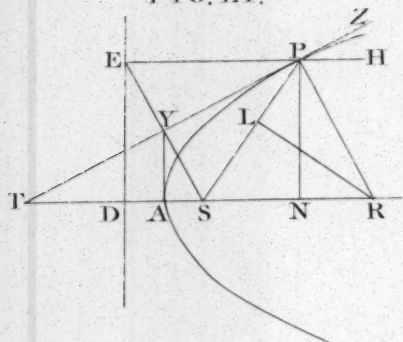


FIG. LIII.

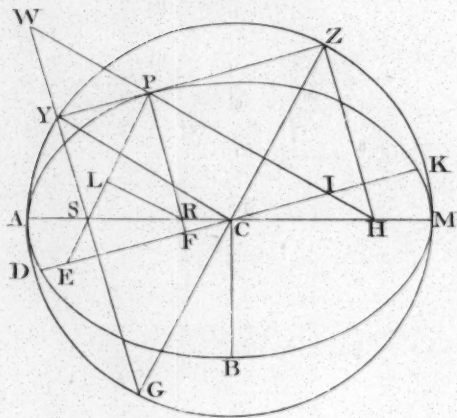


FIG. LVII.

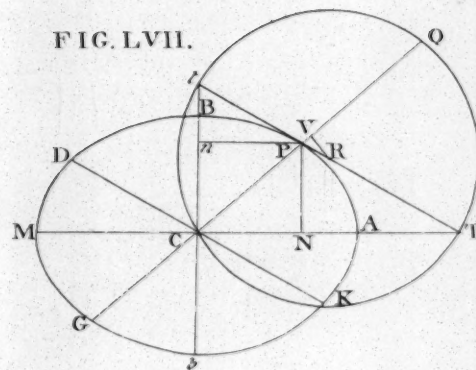


FIG. LIV.

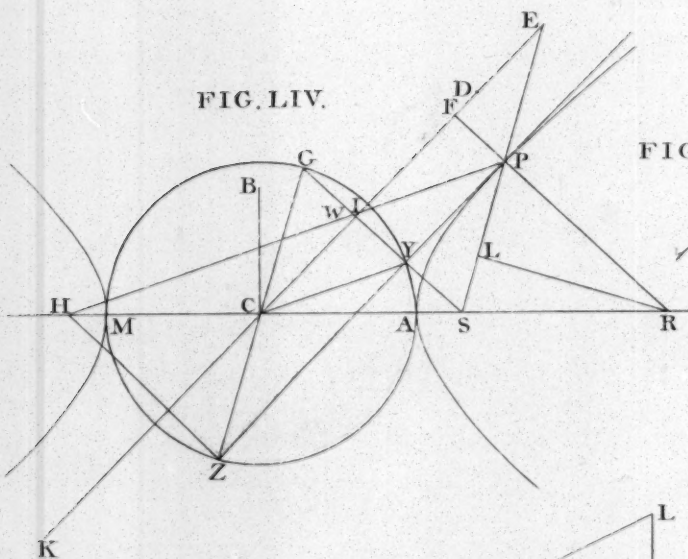


FIG. LII.

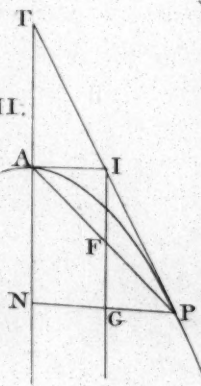


FIG. LVIII.

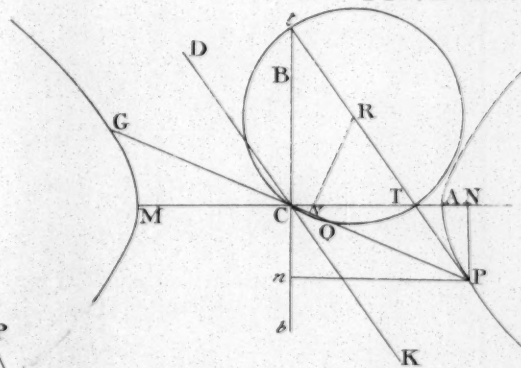


FIG. LV.

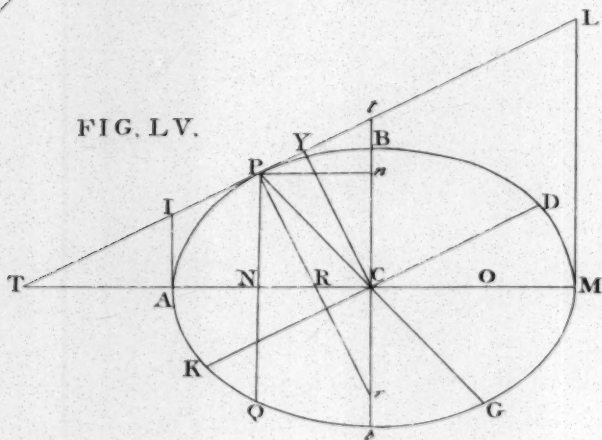
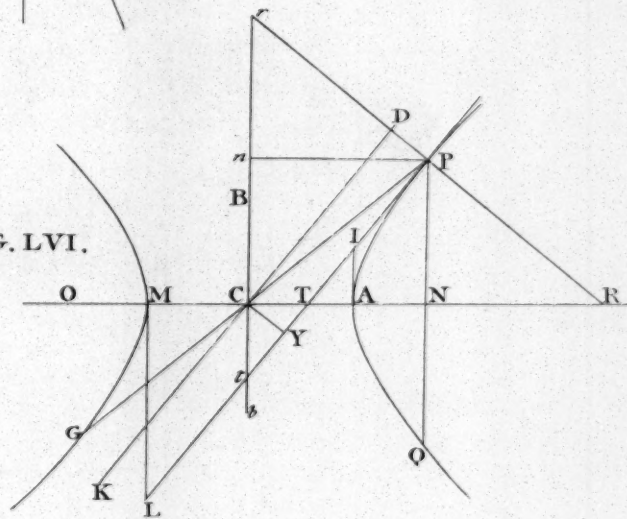


FIG. LVI.



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COR. If a circle be described about any diameter of an ellipse, or an hyperbola, and through any point N in that diameter NR be drawn an ordinate to the circle in the first case, and a tangent to it in the second; the square of NR will be equal to the rectangle ANM ; and if NP be an ordinate to the ellipse or hyperbola drawn through the same point, the square of NR will be to the square of NP as the square of AC is to the square of BC , and NR will be to NP as AC is to BC . FIG. 65,
62.

P R O P. LXI.

If two ellipses or hyperbolas have a common diameter, and an ordinate be drawn to each of the curves through the same point in that diameter; the tangents at the extremities of these ordinates will meet each other in the diameter.

LET $AP, A\mathcal{Q}$ be two ellipses, or two hyperbolas, FIG. 61,
62. having a common diameter AM . Through any point N draw the semi-ordinates $NP, N\mathcal{Q}$; and let CB, CD be the femidiameters parallel to these ordinates. Through the point P draw the tangent PT , meeting the diameter in T , and join $T\mathcal{Q}$, which will be a tangent to the other curve in the point $\mathcal{Q}$: because the tangent PT meets the diameter in T , CN is to CA as CA is to CT , Prop. 49. and therefore $T\mathcal{Q}$ touches the other curve, by the same proposition.

COR. I. If a circle and an ellipse have a common diameter, and a common abscissa; the tangents at the extremities of the ordinates will meet each other FIG. 63.

other in the diameter: for a circle may be considered as an ellipse whose axes are equal. Or it may be deduced from a property of the circle. Let NR be an ordinate of the circle, and NP an ordinate of the ellipse drawn through the same point. Draw RT a tangent to the circle, meeting the diameter in T , and join TP and CR . Then because the triangles CRT , CNR are similar, CN is to CR as CR is to CT , or CN is to CA as CA is to CT ; therefore TP touches the ellipse in the point P .

COR. 2. If the line PNQ drawn through any point N in the diameter of a circle, making any given angle with AM , be bisected in N , and PN be to RN , the ordinate of the circle drawn through the same point, in any given ratio; the points P , Q will be in an ellipse, of which AM is a diameter and PQ an ordinate: for draw BCb through the center parallel to PQ ; take CB and Cb to CA in the given ratio; and describe an ellipse, of which AM , Bb are conjugate diameters. Then because NP , or NQ , is to NR as CB to CA , and PQ is parallel to Bb , PNQ is an ordinate of that ellipse, and the points P , Q are in the curve.

P R O P.

P R O P. LXII.

Let ABM be an ellipse, of which AM is the transverse, and Bb the conjugate axis; from any point F in the conjugate axis let a right line FG , which is equal to the sum or difference of the femiaxes CA, CB , be so placed as to meet the transverse axis in G ; and in FG , produced beyond G when FG is the difference of the femiaxes, let GP be taken equal to CB ; the point P will be in the ellipse. FIG. 65.

FROM the point P draw PN perpendicular to the axis AM ; and through the center C draw CQ parallel to FP , meeting NP produced in Q . Then CQ is equal to FP , which is equal to CA by the construction; therefore the point Q is in the circumference of a circle, of which C is the center and CA radius; and because the triangles PNG, QNC are similar, PN is to QN as PG is to CQ , or as CB to CA ; therefore PN is the semi-ordinate, and P is in the ellipse by the Cor. to the preceding proposition.

COR. Hence, if two right lines AM, Bb , of which AM is the greater, bisect each other at right angles in the point C , and a line FP be taken equal to CA , in which the part PG is taken equal to CB , and whilst the line FP makes one revolution, the point F is always in the line Bb , and G in AM ; the point P will describe an ellipse, of which AM, Bb are the axes.

L.

P R O P.

P R O P. LXIII.

If from the vertices of any two conjugate diameters of an ellipse two ordinates be drawn to the axis; the square of the segments of the axis, between the ordinates and the center, are together equal to the square of the semi-axis; and the squares of the semi-ordinates are together equal to the square of the conjugate semi-axis.

FIG. 63. **L**ET PG, DK be two conjugate diameters of the ellipse; and from the vertices P, D draw the semi-ordinates PN, DL to the axis AM ; the squares of CN, CL are together equal to the square of CA ; and the squares of PN, DL are equal to the square of CB . From the center C , at the distance CA , describe a circle, and let the ordinates NP, LD meet the circumference in R and F ; and join CR, CF . Through the points R, P draw the tangents RT, PT meeting the axis in T ; and because CD is parallel to TP , the triangles CLD, TNP are similar, and

TN is to NP as CL is to LD , and

NP is to NR as LD is to LF ; therefore

TN is to NR as CL is to LF ; and consequently the triangles TNR, CLF are similar, and the angle LCF is equal to NTR , which is equal to CRN ; and CF being equal to CR , the triangles CLF, CNR are equal, and CL is equal to NR ; therefore the sum of the squares of CN, CL is equal to the square of CR , or CA . Secondly, the square of RN is to the square of PN , and the square of FL to the

the square of DL as the square of CA is to the square of CB ; therefore the sum of the squares of RN, FL is to the sum of the squares of PN, DL as the square of CA to the square of CB ; but the sum of the squares of RN, FL , or of CL, FL is equal to the square of CA ; therefore the squares of PN, DL are together equal to the square of CB .

P R O P. LXIV.

The sum of the squares of any two conjugate diameters of an ellipse is equal to the sum of the squares of the axes: and the difference of the squares of any two conjugate diameters of an hyperbola is equal to the difference of the squares of the axes.

FIRST, let ABM be an ellipse, of which PG, DK are any two conjugate diameters; then, the same construction remaining, the square of CP is equal to the sum of the squares of CN, NP , and the square of CD is equal to the sum of the squares of CL, LD ; therefore the squares of CP, CD are together equal to the squares of CN, CL and PN, DL , that is, to the squares of CA, CB ; and the squares of PG, DK are together equal to the squares of AM, Bb . FIG. 63.

Secondly, let PAQ be an hyperbola, of which PG, DK are any two conjugate diameters, PG being the greater of the two; and let AM, Bb be the axes. Join PD cutting the asymptote in l , and draw Pm, Dn perpendicular to the asymptote. Then, Pl being equal to DI , Prop. 37. the angles at n and m being right angles, and the angles at l vertical, FIG. 64.

the triangles Pml , Dnl are equal, and ln is equal to lm : and because the square of CD is less than the squares of Cl , lD , or Cl , lP , by twice the rectangle Clm , or Clm , and the square of CP is greater than the squares of Cl , lP , by twice the rectangle Clm , the difference of the squares of CP , CD will be equal to four times the rectangle Clm . But lm is to lP in a constant ratio of the cosine of the given angle mIP to radius; and the rectangle Clm is to the constant rectangle ClP in the same ratio; therefore the difference of the squares of CP , CD is invariable, and consequently equal to the difference of the squares of CA , CB ; and the difference of the squares of PG , DK is equal to the difference of the squares of AM , Bb .

P R O P. LXV.

The transverse axis is the greatest of all the diameters of an ellipse; and the axes of an hyperbola are the least of all its diameters.

FIG. 63. **L**ET PG be any diameter of the ellipse ABM ; through its vertex P draw PNQ an ordinate to the transverse axis AM , and let it meet the circumference of the circle AEM in R . Because AC is greater than BC ; RN is greater than PN , and the square of RN greater than the square of PN ; therefore the sum of the squares of RN , CN , or the square of CR , is greater than the sum of the squares of PN , CN ; or the square of CP ; and CR , or CA , is greater than CP ; therefore AM is greater than

FIG. 64. PG . Secondly, let PG be any diameter of the hyperbola PAQ , and draw PNQ an ordinate to the axis MA . Then, CNP being a right angled triangle,

gle, CP is greater than CN , which is greater than CA ; therefore PG is greater than AM . In the same manner it may be proved that, if DK be any diameter of the conjugate hyperbola, CD will be greater than CB , and DK greater than Bb .

COR. Those diameters of the hyperbola which are nearer to the axis are less than those which are more remote: for as PN decreases, CN decreases, Cor. 5. Prop. 6. and therefore CP decreases.

P R O P. LXVI.

Those diameters which are nearer to the transverse axis of an ellipse are greater than those which are more remote; the conjugate axis is the least of all the diameters; and any two diameters of the ellipse or hyperbola, which make equal angles with either of the axes, are equal.

LET ABM be an ellipse; describe a circle FIG. 63.
 having the axis AM for its diameter; and let the ordinates to the transverse axis AM be produced to meet the circumference of the circle. Because the square of RN is to the square of PN in a constant ratio, the difference of the squares of RN , PN will be to the square of RN in a constant ratio; and if the line RPN be supposed to move from A to C , RN will increase till it becomes equal to EC , or CA , whilst CN decreases; therefore the square of PN , and the difference of the squares of RN , PN , which is equal to the difference of the squares of CR , CP , will increase from A to C ; and, CR being constant,

constant, the square of CP will decrease; and therefore CP will decrease, till it becomes equal to CB , when it will be the least.

FIG. 63,

64.

Secondly, let PCQ be any diameter of the ellipse or hyperbola; draw PNQ an ordinate to either of the axes, and join CQ . Then, NQ being equal to NP , and CN common to the two right angled triangles CNQ , CNP , the triangles are equal; therefore CQ is equal to CP , and the angle QCN is equal to the angle PCN .

COR. Hence, if the semidiameters CP , CD be equal, and the angle PCQ be bisected by the line CA , the position of the axis will be determined.

P R O P. LXVII.

The two diameters of an ellipse, which bisect the right lines joining the vertices of the axes, are equal and conjugate diameters.

FIG. 66.

LET AM , Bb be the axes of an ellipse; join AB , BM , and draw the diameters GP , KD , bisecting the lines AB , BM in the points N and I . Because the lines BA , BM are bisected by the diameters GP , KD , they are ordinates to these diameters, Cor. 1. Prop. 33. and because AM is bisected in C , and AB in N ; CN is parallel to MB ; therefore GP , KD are conjugate diameters: and the angle ACB being a right angle, it will be in a semicircle, of which AB is the diameter, and N the center; therefore NA will be equal to NC , and the angle NCA equal to the angle NAC , which is equal to the alternate angle ACK ; and therefore the diameters PG , KD are equal, by the preceding proposition.

P R O P.

FIG. LIX.

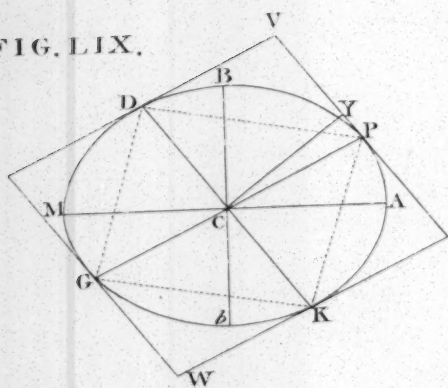


FIG. LX.

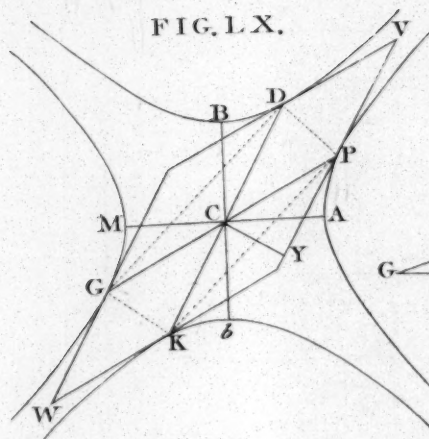


FIG. LXV.

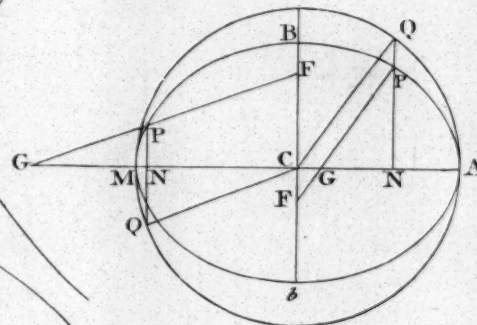


FIG. LXI.

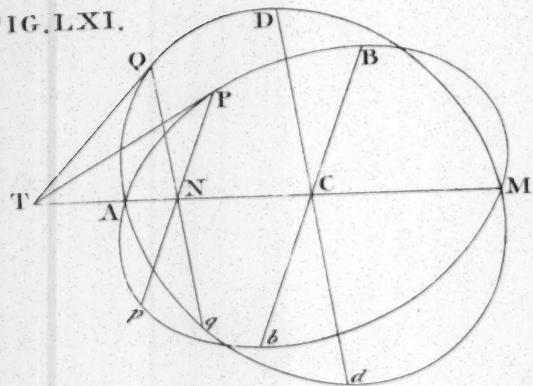


FIG. LXII.

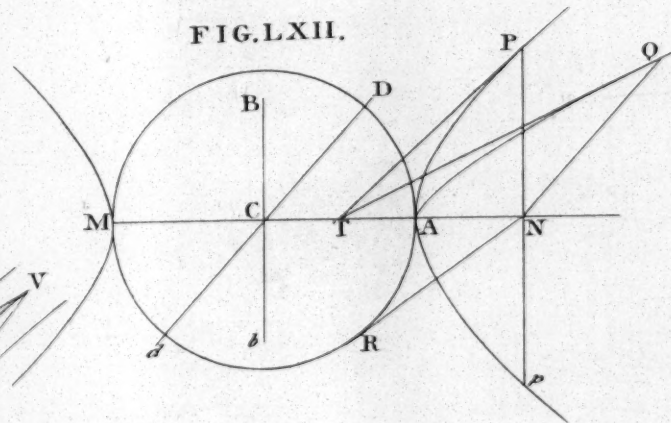


FIG. LXIV.

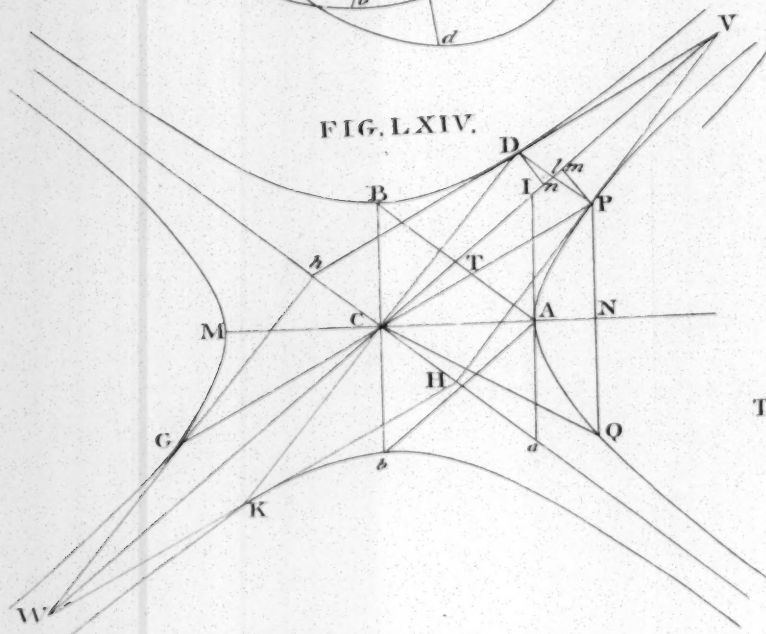
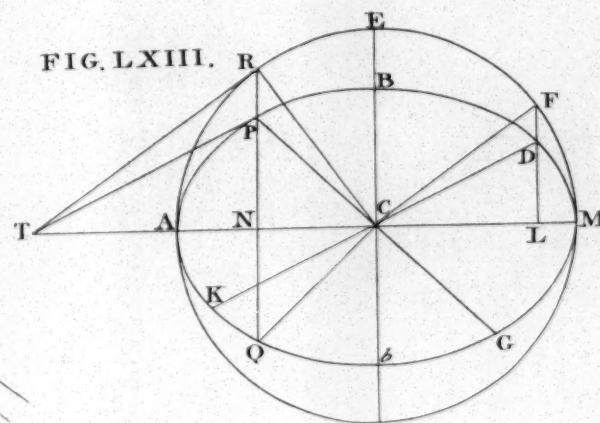


FIG. LXIII.



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PROP. LXVIII. PROB. IX.

To find the axes of a given conic section.

FIRST, let the section be an ellipse or an hyperbola, and find any two diameters, by Cor. 4. Prop. 33. cutting each other in the point C : from the center C , at the distance CP , which is the greater semidiameter in the hyperbola, and which is greater than the lesser semidiameter in the ellipse, but less than the greater, describe a circle; which will cut the curve, or the opposite curves, in the points P and G ; and because the ellipse and hyperbola have each of them another diameter equal to PG , it will also cut the curve, or opposite curves, in two other points Q and K . Join CQ , and draw CA bisecting the angle PCQ , which will be one of the axes, Cor. Prop. 66. Join PQ , which will be bisected by the line CA in N ; it is, therefore, an ordinate to the axis; and if BCb be drawn through C parallel to PQ , meeting the ellipse in the points B, b , it will be the other axis; the length of which, in the hyperbola, may be determined in the following manner; from the center C , at the distance CA , describe a circle; from the point N draw NR a tangent to the circle, and take CB, Cb each of them a fourth proportional to the lines RN, PN and CA ; and Bb will be the conjugate axis, Cor. Prop. 60.

FIG. 67,
68.

FIG. 68.

Secondly, let the section be a parabola, of which find any diameter BC ; and if it bisects its ordinates at right angles, it is the axis; if it does not bisect them at right angles, through any point C draw QCP perpendicular to BC , meeting the parabola in the points Q, P ; bisect QP in the point N , and draw NA

FIG. 69.

NA parallel to CB , which will be the axis: for NA is a diameter which bisects QP ; therefore QP is an ordinate to that diameter; and because it is perpendicular to NA , NA is the axis.

P R O P. LXIX.

If two ellipses or two hyperbolas have a common axis, and an ordinate be drawn through the same point in the axis to each of the curves; the areas included between the common abscissa, the ordinates, and the two curves, also the whole areas of the ellipses will be to each other as the conjugate axes.

FIG. 70,
71.

LET AP , AQ be two ellipses, or two hyperbolas; take any abscissa AN , which is not greater than half the axis of the ellipse, and draw the ordinates NP , NQ . The areas ANP , ANQ are to each other as the conjugate axes. Let the abscissa AN be divided into any number of equal parts, AE , EF , FG , GN ; through the points E , F , G draw the ordinates ERI , FSK , GTL , and complete the parallelograms, AR , AI , ES , EK , &c. also from the points I , K , L draw Ii , Kk , Ll parallel to AN . Then it is evident that the difference between the circumscribed parallelograms AI , EK , FL , GQ and the inscribed parallelograms Ei , Fk , Gl is equal to the parallelogram GQ ; and if parallelograms be inscribed, in the same manner, in the other figure APN , the difference between these and the circumscribed parallelograms would be equal to the parallelogram GP ; therefore the differences between each series of parallelo-

parallelograms and the areas AQN , APN will be less than the parallelograms GQ , GP ; and because the parallelogram GP is to the parallelogram GQ as NP is to NQ , and each parallelogram in the figure APN is to the corresponding parallelogram in the figure AQN in the same ratio; the sum of all the parallelograms in the figure APN is to the sum of all in the figure AQN as NP is to NQ ; and the area APN will be to the area AQN as the sum of the parallelograms in APN to the sum in AQN : for if not, let the parallelograms APN be to the parallelograms AQN as the area APN to some space X greater or less than the area AQN . First, let the space X be greater; then if the bases AE , EF , FG , GN be continually bisected, and the parallelograms completed as above, the parallelograms GP , GQ , will be diminished in the ratio of two to one at each bisection, and the difference between each series of parallelograms and the areas APN , AQN will be diminished more than half; therefore the difference between the circumscribed parallelograms AQN and the area AQN may be made less than any given space; let it be less than the difference between the area AQN and the space X , and the circumscribed parallelograms AQN will be less than the space X ; but the sum of the parallelograms APN is to the sum of the parallelograms AQN as the area APN is to the space X , and the sum of the parallelograms APN is greater than the area APN ; therefore the sum of the parallelograms AQN is greater than X ; which is impossible. Secondly, let the parallelograms APN be to the parallelograms AQN as the area APN to some space X less than the area AQN , and let the difference between the inscribed parallelograms AQN and the area AQN be made less than the difference between the area AQN and X , then the parallelo-

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NA parallel to CB , which will be the axis: for NA is a diameter which bisects QP ; therefore QP is an ordinate to that diameter; and because it is perpendicular to NA ; NA is the axis.

P R O P. LXIX.

If two ellipses or two hyperbolas have a common axis, and an ordinate be drawn through the same point in the axis to each of the curves; the areas included between the common abscissa, the ordinates, and the two curves, also the whole areas of the ellipses will be to each other as the conjugate axes.

FIG. 70,
71.

LET AP , AQ be two ellipses, or two hyperbolas; take any abscissa AN , which is not greater than half the axis of the ellipse, and draw the ordinates NP , NQ . The areas ANP , ANQ are to each other as the conjugate axes. Let the abscissa AN be divided into any number of equal parts, AE , EF , FG , GN ; through the points E , F , G draw the ordinates ERI , FSK , GTL , and complete the parallelograms, AR , AI , ES , EK , &c. also from the points I , K , L draw Ii , Kk , Ll parallel to AN . Then it is evident that the difference between the circumscribed parallelograms AI , EK , FL , GQ and the inscribed parallelograms Ei , Fk , Gl is equal to the parallelogram GQ ; and if parallelograms be inscribed, in the same manner, in the other figure APN , the difference between these and the circumscribed parallelograms would be equal to the parallelogram GP ; therefore the differences between each series of parallelo-

parallelograms and the areas AQN , APN will be less than the parallelograms GQ , GP ; and because the parallelogram GP is to the parallelogram GQ as NP is to NQ , and each parallelogram in the figure APN is to the corresponding parallelogram in the figure AQN in the same ratio; the sum of all the parallelograms in the figure APN is to the sum of all in the figure AQN as NP is to NQ ; and the area APN will be to the area AQN as the sum of the parallelograms in APN to the sum in AQN : for if not, let the parallelograms APN be to the parallelograms AQN as the area APN to some space X greater or less than the area AQN . First, let the space X be greater; then if the bases AE , EF , FG , GN be continually bisected, and the parallelograms completed as above, the parallelograms GP , GQ , will be diminished in the ratio of two to one at each bisection, and the difference between each series of parallelograms and the areas APN , AQN will be diminished more than half; therefore the difference between the circumscribed parallelograms AQN and the area AQN may be made less than any given space; let it be less than the difference between the area AQN and the space X , and the circumscribed parallelograms AQN will be less than the space X ; but the sum of the parallelograms APN is to the sum of the parallelograms AQN as the area APN is to the space X , and the sum of the parallelograms APN is greater than the area APN ; therefore the sum of the parallelograms AQN is greater than X ; which is impossible. Secondly, let the parallelograms APN be to the parallelograms AQN as the area APN to some space X less than the area AQN , and let the difference between the inscribed parallelograms APN and the area AQN be made less than the difference between the area AQN and X ; then the parallelo-

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grams AQN will be greater than X , but the inscribed parallelograms APN are less than the area APN ; therefore the parallelograms AQN are less than X ; which is impossible. Therefore the area APN is to the area AQN as the parallelograms APN to the parallelograms AQN , or as NP is to NQ , that is, as the conjugate axes Prop. 60.

FIG. 70. If the sections be two ellipses, and the abscissa MN be greater than half the axis, the area ACB is to the area ACD as CB to CD ; and by division, the area $CNPB$, is to the area $CNQD$ as CB to CD ; therefore by composition, the area MPN is to the area MQN , and the area MBA to the area MDA as CB is to CD , and consequently the whole areas $MBAb$, $MDAd$ are in the same ratio.

COR. 1. If a circle be described about the transverse axis of an ellipse; the area of the circle will be to the area of the ellipse, as the transverse axis is to the conjugate axis.

FIG. 70, 72. COR. 2. The area of an ellipse is equal to that of a circle, whose diameter is a mean proportional between the two axes: for let $ABMb$ be an ellipse, and let $ADMd$ be a circle having the transverse axis for a diameter; let HV be a mean proportional between AM , Bb , and describe the circle having HV for a diameter. Then, because AM , HV , Bb are continual proportionals, the square of AM is to the square of HV as AM is to Bb , that is, as the area of the circle $ADMd$ to the area of the ellipse $ABMb$; and the square of AM is to the square of HV as the circle $ADMd$ to the circle HXV ; therefore the area of the circle HXV is equal to the area of the ellipse $ABMb$.

COR. 3. The areas of any two ellipses are to each other as the rectangles under their axes: for the area of the ellipse $ABMb$ is to the circle HXV as the

the rectangle under AM, Bb is to the square of HV , that is, in a ratio of equality; and alternately, the area of the ellipse $ABMb$ is to the rectangle under AM, Bb as the circle HXV is to the square of HV ; but all circles are as the squares of their diameters; therefore the area of the ellipse is to the rectangle under the axes in a given ratio.

P R O P. LXX.

If two parabolas have a common axis, and an ordinate be drawn through the same point in the axis to each of the curves; the areas of the parabolas, included between the common abscissa, the ordinates, and the two curves, are to each other in a subduplicate ratio of the latera recta,

LET $AP, A\mathcal{Q}$ be two parabolas, having a com- FIG. 73.
mon axis AN ; through any point N in the axis draw the ordinates $NP, N\mathcal{Q}$; and let L and M be the latera recta of the two parabolas $AP, A\mathcal{Q}$. Then the square of NP is to the square of $N\mathcal{Q}$ as the rectangle under AN and L to the rectangle under AN and M , or as L is to M ; therefore NP is to $N\mathcal{Q}$ in the subduplicate ratio of L to M ; and the ordinates $NP, N\mathcal{Q}$ being to each other in a constant ratio, it may be proved, in the same manner as in the preceding proposition, that the areas $APN, A\mathcal{Q}N$ are to each other in the same ratio,

P R O P. LXXI.

If any ordinate and abscissa of a parabola be completed into a parallelogram; the area of the parabola, included between the ordinate and the curve, is to the parallelogram as 2 to 3.

FIG. 74. **L**ET AN be any diameter of the parabola, and PQ an ordinate to that diameter; through the point A draw BC parallel to PQ ; and through the points P, Q draw PB, QC parallel to NA . The area of the parabola PAQ will be to the parallelogram $PBCQ$ as 2 to 3. Join PA, AQ ; and through the points P, Q draw the tangents PT, QT , meeting the diameter in T : through the points E, G draw the diameters ED, GK , which will bisect the lines PA, AQ in D, K , Cor. 2. Prop. 34. and through the vertices draw the tangents RL, MV ; join PF, FA , and AH, HQ . Then, NA being equal to AT , and PQ equal to twice EG , the triangle PAQ will be double the triangle TEG ; for the same reason the triangles PFA, AHQ will be double the triangles ERL, GMV ; therefore the inscribed figure $PFAHQ$ will be double the external figure $TRLAMV$, and the same proportion holds whatever be the number of triangles inscribed: but, by proceeding in this manner, the difference between the inscribed figure $PFAHQ$ and the area PAQ , and the difference between the external figure and the area $TPAQ$ will each of them become less than any given space: for the triangle PAQ being half of the parallelogram $PBCQ$, it is greater than half of the area PAQ ; and, for the same reason, the

tri-

triangles PFA , AHQ are each of them greater than half of the areas PFA , AHQ ; and PE being equal to ET , the triangle PEA is equal to the triangle EAT , and the triangle TEG is half of the two triangles TPA , TQA , and therefore more than half of the external area $TPAQ$; and, for the same reason, the triangles ERL , GMV are more than half of the areas $EPFA$, $GAHQ$; therefore, by inscribing triangles as above, the difference between each rectilinear figure and the parabolic area is diminished more than half at each operation, and therefore may be made less than any given space; and the parabolic area PAQ will be double the area $TPAQ$; for if it were greater than double that area by any given space S , a series of triangles might be inscribed in the area PAQ , which would differ from it by a space less than S , and this series of triangles would be more than double the area $TPAQ$, and consequently more than double the corresponding figure inscribed in the area $TPAQ$; which is impossible. If the area PAQ were less than double the area $TPAQ$, this area would be greater than half the area PAQ by some space S ; and therefore since a series of triangles might be inscribed in the area $TPAQ$, which would differ from it by a space less than S , this series would be greater than half of the area PAQ , and therefore greater than half of the series of corresponding triangles in the area PAQ ; which is impossible. Therefore it follows that the area PAQ is neither more nor less than double the area $TPAQ$, and consequently the area PAQ is to the whole triangle PTQ as 2 to 3. But the triangle PTQ is equal to the parallelogram $PBCQ$; therefore the area PAQ is to the parallelogram $PBCQ$ as 2 to 3.

COR. Hence, the area $PFAN$ being two thirds
be

of the parallelogram BN , the external area $PBAF$ will be one third of the parallelogram, or two thirds of the triangle PAB ; therefore the area PFA , between the ordinate PA and the curve, is a third part of the triangle PBA .

P R O P. LXXII.

FIG. 75.

If the right lines PN , QB be drawn from the vertices of any two semidiameters of an hyperbola parallel to the asymptote CA , meeting the other asymptote CB in the points N , B ; the hyperbolic sector CPQ will be equal to the hyperbolic trapezium $PNBQ$.

LET the semidiameter CQ meet the line PN in D ; and because the triangles CPN , CQB are equal, Cor. 3. Prop. 36. if from these be taken the common part CDN , the triangle CPD will be equal to the trapezium $NDQB$, and if to these be added the figure PDQ , the hyperbolic sector PCQ will be equal to the hyperbolic trapezium $PNBQ$.

P R O P.

P R O P. LXXIII.

If from the center of an hyperbola the segments CD, CE, CF of an asymptote be taken in continued proportion, and the right lines DP, EA, FQ be drawn parallel to the other asymptote, meeting the hyperbola in the points P, A, Q ; the hyperbolic areas $PDEA, AEFQ$ will be equal. FIG. 76.

JOIN PQ , and let it meet the asymptotes in R and T . Through the point A draw the tangent IAG , meeting the asymptotes in I and G , and draw the semidiameters CP, CA, CQ . Because AE is parallel to IC , and GA is equal to AI , Cor. 1. Prop. 35. GE is equal to EC ; and because QF, PD, RC are parallel, and TQ is equal to PR , TF is equal to CD ; therefore TF is to GE as CE is to CF , or as FQ to EA , Cor. 2. Prop. 36. and alternately, TF is to FQ as GE is to EA , and the angles at F and E being equal, the triangles TFQ, GEA are similar, and TQ is parallel to GA , or QP is parallel to GA ; it is, therefore, an ordinate to the diameter CA , which is bisected in N ; and as the diameter bisects all lines which are parallel to PQ , and are terminated by the hyperbola, it will bisect the area QAP ; and if the areas PAN, NAQ be taken from the equal triangles PCN, NCQ , the hyperbolic sectors CPA, CAQ will be equal; and therefore, by the preceding proposition, the areas $PDEA, AEFQ$ are equal.

COR. 1. If CL be taken a third proportional to CE, CF , and LM be drawn parallel to the asymptote

tote CI ; it may be proved, in the same manner, that the areas $AEFQ$, $QFLM$ are equal.

COR. 2. Hence, if the segments of the asymptote be taken in continued proportion, the areas, beginning from the first line DP , will be in arithmetical proportion.

DEFINITIONS.

XXVIII. A circle is said to touch a conic section in any point, when the circle and the conic section have a common tangent in that point.

XXIX. If a circle touches a conic section in any point, so that no other circle can be drawn between this circle and the conic section, it is said to have the same curvature with the section in the point of contact, and it is called the circle of Curvature.

LEMMA III.

FIG. 77. If two tangents PT , VT be drawn at the extremities of any chord of a circle, and from any point Q in the circumference a chord Qq be drawn parallel to one of the tangents TP , cutting the chord PV in the point N , and from the points Q , q the lines QH , qb be drawn parallel to the other tangent TV , meeting the chord PV in H and b ; the square of QN will be equal to the rectangle under PN , VH ; and the square of qN will be equal to the rectangle under PN , Vb .

JOIN

FIG. LXVI.

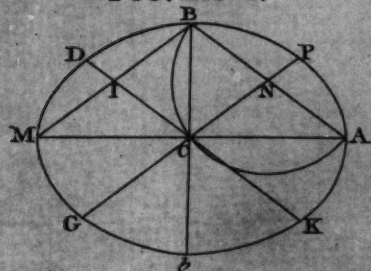


FIG. LXVIII.

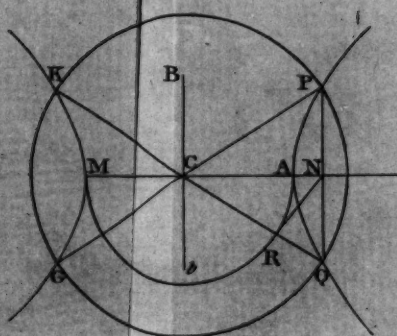


FIG. LXVII.

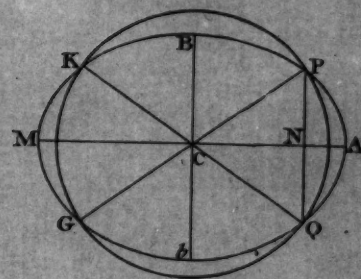


FIG. LXIX.

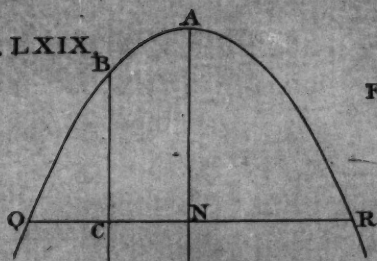


FIG. LXXII.

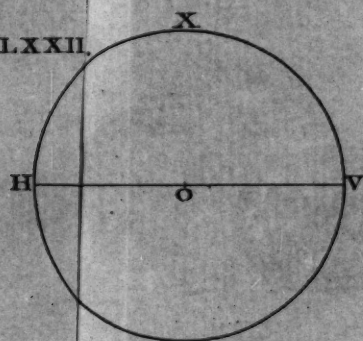


FIG. LXX.

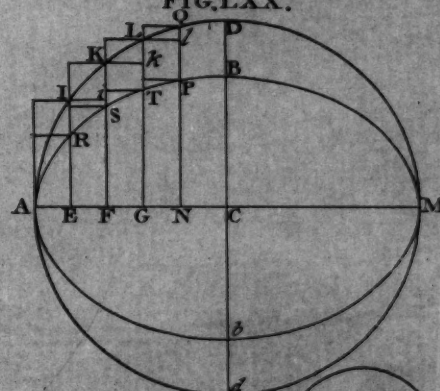


FIG. LXXIV.

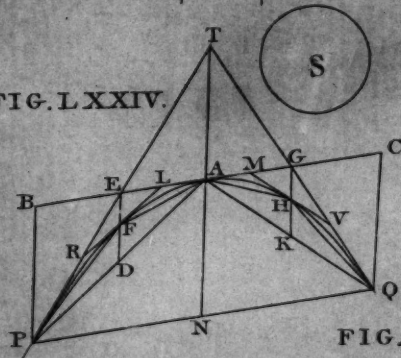


FIG. LXXVI.

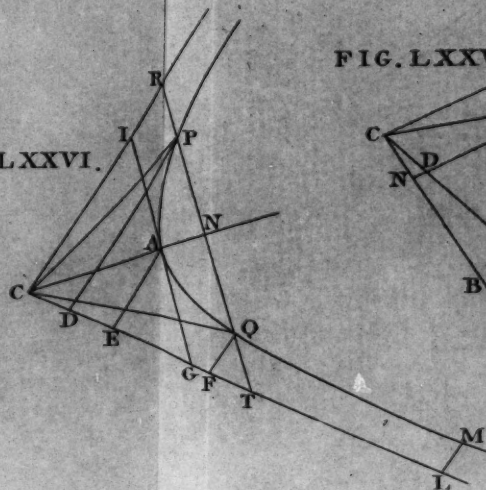


FIG. LXXV.

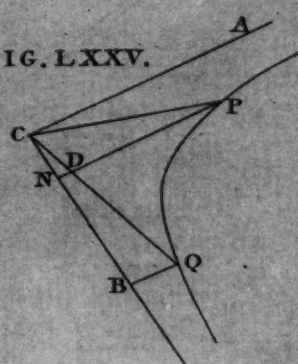


FIG. LXXI.

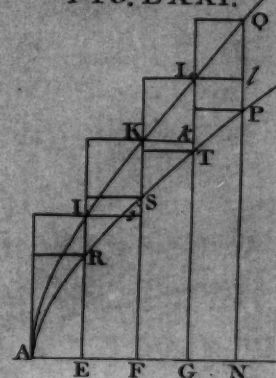
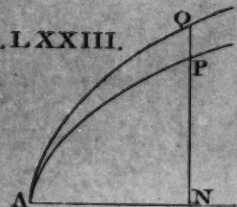


FIG. LXXIII.



JOIN QN , QH , also qP , qV ; and because the lines QN , QH are parallel to TP , TV , the triangles TPV , QNH are similar; therefore QN is equal to QH ; the angle QHN is equal to the angle QNH ; and the angle QHV equal to the angle QNP ; and the angle PQN being equal to the alternate angle QPT , which is equal to the angle QVP in the alternate segment, the triangles QPN , QVH are similar; therefore PN is to QN as QH , or QN , is to VH , and the rectangle under PN , VH is equal to the square of QN . In the same manner it may be proved, that the triangles PqN , qVh are similar; therefore PN is to qN as qh , or qN , is to Vh , and the rectangle under PN , Vh is equal to the square of qN .

COR. If the line Qq , which is always parallel to TP , be supposed to move from V to P ; the points H , N , h , and the points Q , q will continually approach to P , in which point they will all coincide.

P R O P. LXXIV.

If a circle touches a conic section in any point, and cuts off from the diameter which passes through that point a segment greater than its parameter, a part of the circumference on each side of the point of contact will be wholly without the conic section; and if it cuts off from the diameter a segment less than its parameter, a part of the circumference on each side of the point of contact will be wholly within the conic section.

FIG. 78. *Case 1.* **L**ET the conic section be a parabola, of which PV is any diameter; let PR be a common tangent to the parabola and the circle in the point P ; take PR equal to the parameter of that diameter, and through the point R draw RL parallel to PV ; let the chord Qq , which is parallel to RP , be produced to meet RL in L , and from the points Q, q draw QH, qh parallel to the tangent TV . First, let the circle cut off a segment PV greater than the parameter, and take VB equal to PR , or NL . Then let the line Qq be supposed to move from V to P , and when the point H comes to B , VH will be equal to VB , or NL ; and the rectangle under PN, VH , or the square of NQ , will be equal to the rectangle under PN, NL , which is equal to the square of the semi-ordinate of the parabola, Prop. 43. therefore NQ will be equal to the semi-ordinate, and Q is a point in the parabola.

But

But when the point H is any where between B and P , VH will be greater than NL ; therefore NQ , and consequently Nq will be greater than the semi-ordinate, and the arc of the circle QPq will be wholly without the parabola. Secondly, let PV be less than PR , and take Vb equal to PR . When h comes to b , Vh will be equal to NL ; therefore Nq will be equal to the semi-ordinate, and q is a point in the parabola; but when h is any where between b and P , Vh will be less than NL ; therefore Nq , and consequently NQ will be less than the semi-ordinate, and the arc QPq will be wholly within the parabola.

Case 2. Let the section be an ellipse, of which PG is any diameter, and take PR in the tangent at P equal to the parameter of that diameter; join RG , and let the chord Qq meet RG in the point L . First, let the chord PV be greater than the parameter, and take Vb equal to PR ; then, NL being less than PR , VH will be equal to NL before H comes to B , when NQ will be equal to the semi-ordinate, and Q will be a point in the ellipse; when H is any where between this point and P , VH will be greater than NL ; therefore NQ , and consequently Nq will be greater than the semi-ordinate, and the arc QPq will be wholly without the ellipse. Secondly, if PV be less than the parameter, take Vb equal to PR ; from P draw PO parallel to RL , meeting the chord Qq in O ; then OL will be equal to PR ; and as the point N approaches to P , the point O will approach to N ; and as h moves from b towards P , bh increases, and NO decreases; therefore bh will be equal to NO before h comes to P , and Vh will be equal to LN ; therefore Nq will be equal to the semi-ordinate; from this point to P , bh will be greater than NO , and Vb less than NL ; there-

FIG. 79.

fore Nq , and consequently NQ will be less than the semi-ordinate, and the arc Qp will be wholly within the ellipse.

FIG. 80.

Case 3. Let the section be an hyperbola, of which PG is any diameter, in the tangent at P take PR equal to the parameter; join RG , and produce the chord Qq till it meet GR in L ; and NL will be greater than PR . First, let the chord PV be greater than the parameter; take VB equal to PR , and from P draw PO parallel to RL . When the point H is between B and P , but nearer to B , BH will be less than NO ; and as H approaches to P , BH increases, and NO decreases; therefore BH will be equal to NO before H comes to P , and VH will be equal to NL ; therefore NQ will be equal to the semi-ordinate of the hyperbola; from this point to P , BH being greater than NO , VH will be greater than NL ; therefore NQ , and consequently Nq will be greater than the semi-ordinate, and the arc Qp will be wholly without the hyperbola. Secondly, let PV be less than the parameter, and take Vb equal to PR . When h comes to b , Vh will be equal to PR ; it will, therefore, be equal to NL before h comes to b , when Nq will be equal to the semi-ordinate of the hyperbola; from this point to P , Vh will be less than NL ; therefore Nq , and consequently NQ will be less than the semi-ordinate, and the arc Qp will be wholly within the hyperbola.

COR. 1. If a circle touches a conic section, and cuts off from the diameter which passes through the point of contact a segment equal to its parameter, no other circle can be drawn between this circle and the conic section: for if a greater circle be described, it will cut off from the diameter a segment greater than its parameter, and a part of the cir-

circumference on each side of the point of contact will be wholly without the conic section, it will also be without the former circle; and if a less circle be described, it will cut off from the diameter a segment less than its parameter; it will, therefore, be within the conic section on each side of the point of contact, and it will fall within the former circle.

COR. 2. The chord, which the circle of curvature cuts off from the diameter of a parabola, is equal to four times the distance of the vertex of that diameter from the focus.

COR. 3. The chord, which the circle of curvature cuts off from a diameter of an ellipse or an hyperbola, is a third proportional to that diameter and its conjugate.

COR. 4. If two conic sections have the same parameter, and the ordinates of each make the same angle with the diameter, they will have the same circle of curvature.

PROP.

P R O P. LXXV.

The circle of curvature at the vertex of the transverse axis of an ellipse or hyperbola, or at the vertex of the axis of a parabola, falls wholly within the conic section: but the circle of curvature at the vertex of the conjugate axis of an ellipse falls wholly without the ellipse.

- FIG. 78. **T**HE same construction remaining, the chord
 79. PV at the vertex of an axis will be perpendicular to the tangent; it will, therefore, be a diameter of the circle; and Qq being perpendicular to PV , it will be bisected in N , and QH, qh will coincide with QN, qN ; and the squares of QN, qN will each of them be equal to the rectangle PNV ; and if the section be a parabola or an hyperbola,
 FIG. 78. VN will be less than VP , or PR , and consequently less than NL ; therefore NQ will be less than the semi-ordinate, and the circle will fall wholly within the section. If the section be an ellipse,
 FIG. 79. PN is to NO as PG is to PR ; therefore PN will be greater or less than NO , and VN less or greater than NL , according as PG is greater or less than PR . At the vertex of the transverse axis PG is greater than PR ; therefore VN is less than NL , and NQ is less than the semi-ordinate; but at the vertex of the conjugate axis PG is less than PR ; therefore VN is greater than NL , and NQ is greater than the semi-ordinate. Therefore in the former case the circle falls wholly within, and in the latter, without the ellipse.

P R O P.

P R O P. LXXVI.

The circle of curvature at the vertex of any diameter of a conic section, which is not an axis, cuts the section in that point; it also cuts it in another point, which may be determined.

Case 1. **L**ET the section be a parabola; and, the same construction remaining, draw PM parallel to the tangent TV , meeting the circle in the point M . Then, MP being parallel to qh , if q be any where in the arc PM , between P and M , Vh will be greater than VP , or NL ; therefore Nq will be greater than the semi-ordinate. When q is at M , Vh will be equal to VP , or NL , and M is a point in the parabola. If q be any where between M and V , Vh will be less than NL , and Nq less than the semi-ordinate; and if Q be any where between P and V , VH will be less than NL , and NQ less than the semi-ordinate. Therefore the circle cuts the parabola in the points P and M ; the arc PqM is without, and the arc PQM is within the parabola. FIG. 78.

Case 2. If the section be an ellipse; draw PW parallel to the tangent TV . Then it is evident, that, if q be any where in the arc PW , Vh will be greater than PR , and consequently greater than NL ; therefore Nq is greater than the semi-ordinate, and the arc PqW is without the ellipse. In the tangent TV take TE to TV as the diameter PG to the parameter PR , and draw PE cutting the circle in M ; then the arc PQM will be within the ellipse, and the circle will cut the ellipse in M : for take any FIG. 79.

any point Q between P and M ; join PQ , and produce it till it meet TV in F , and let HQ meet the tangent TP in I . Because IH is parallel to TV , IH is to TV as PI is to PT , or as IQ to TF ; and alternately, IH is to IQ as TV is to TF ; but, NQ being parallel to PI , PH is to PN as IH to IQ , as TV to TF , and PN is to NO as GP to PR , as TE to TV ; therefore PH is to NO as TE is to TF . Hence, whilst TF is less than TE , NO will be less than PH , and consequently NL greater than VH , and the semi-ordinate greater than NQ . If Q be at M , NO will be equal to PH , and NL equal to VH ; therefore M is a point in the ellipse. But if TF be greater than TE , NL will be less than VH , and the semi-ordinate less than NQ ; therefore the circle cuts the ellipse in M .

If F be on the other side of V , or q be any where between V and W , it may be shown in the same manner, that Vh will be greater than NL , and consequently Nq greater than the semi-ordinate. Therefore the circle cuts the ellipse in the points P and M ; the arc PWM is without, and the arc PQM is within the ellipse.

FIG. 80. *Case 3.* If the section be an hyperbola, VH being less than VP , or PR , and consequently less than NL , NQ is less than the semi-ordinate. In VT produced take TE to TV as PG is to PR ; join EP , and produce it to meet the circle in M . Take any point q between P and M ; join qP , and produce it to meet TE in F ; and let the tangent TP meet qh in I . Then, qh being parallel to VF , hI is to TV as PI is to PT , as Iq to TF ; and alternately, hI is to Iq as TV is to TF ; but hP is to PN as hI to Iq , as TV to TF , and PN is to NO as PG to PR , as TE to TV ; therefore hP

hP is to NO as TE is to TF . Hence, whilst TF is less than TE , NO will be less than hP , and consequently NL less than Vh ; therefore Nq is greater than the semi-ordinate. If q be at M , TF will be equal to TE , and Vh will be equal to NL ; therefore M is a point in the hyperbola. Draw PW parallel to TV ; then if q be any where between M and W , TF will be greater than TE , and NL greater than Vh ; therefore Nq is less than the semi-ordinate; but when q comes to W , Vh is equal to VP , or PR , which is less than NL , and from W to V , Vh will be less than VP . Therefore the circle cuts the hyperbola in the points P and M ; the arc PqM is without, and the arc PQM is within the hyperbola.

P R O P. LXXVII.

The chord of the circle of curvature, which is drawn from the point of contact through the focus of a parabola, is equal to that which is cut off from the diameter; and half the radius of the circle is a third proportional to the perpendicular drawn from the focus upon the tangent, and the distance of the point of contact from the focus.

LET PV be the chord which is cut off from the diameter; draw PSW through the focus, meeting the circle in W ; and draw the diameter PR ; join VW , RW ; bisect PR in O ; and draw SY from the focus perpendicular to the tangent. Then,

O

the

FIG. 81.

the angle SPY being equal to the angle VPZ , Prop. 25. the angles in the alternate segments will be equal, that is, the angle PVW equal to the angle PWW , and PW is equal to PV . Secondly, the triangles RPW , SPY being similar, RP is to PW , or $4SP$, as SP is to SY ; therefore half PO is to SP as SP is to SY .

COR. 1. Hence the radius of curvature is equal to $\frac{2SP^2}{SY}$.

COR. 2. Because the radius of curvature varies as the square of SP directly, and as SY inversely, and SP varies as the square of SY , Cor. 1. Prop. 45. the radius of curvature will vary as the cube of the perpendicular SY , or as the cube of the normal, Cor. 3. Prop. 45.

P R O P. LXXVIII.

The radius of the circle of curvature at the vertex of any diameter of an ellipse, or an hyperbola, is a third proportional to the perpendicular drawn from the center upon the tangent, and the conjugate semidiameter; and the chord which is drawn from the point of contact through the focus is a third proportional to the transverse axis, and the conjugate diameter.

FIG. 82,
83. **L**ET PV be the chord which is cut off from the diameter; draw the diameter of the circle PR , and from the center O draw OT perpendicular to PV , which will bisect PV in T ; draw the conju-

conjugate diameter DCK , cutting PR in F , and PF will be equal to the perpendicular drawn from the center C upon the tangent; draw the chord PW through the focus S , and let it meet the conjugate diameter in E ; and join RW . Because the triangles PFC , PTO are similar,

PF is to PC as PT is to PO , and

PC is to CD as CD is to PT ; therefore

PF is to CD as CD is to PO .

Secondly, because the triangles PEF , PRW are similar,

PW is to PR as PF to PE , or as $2PF$ to $2PE$, and

PR is to $2CD$ as $2CD$ is to $2PF$; therefore

PW is to $2CD$ as $2CD$ is to $2PE$, or $2AC$.

COR. 1. Hence the radius of curvature is equal to $\frac{CD^2}{PF}$; and the chord which is drawn through the

focus is equal to $\frac{2CD^2}{AC}$.

COR. 2. Because CD is to AC as CB to PF , Prop. 56. the square of CD will vary inversely as the square of PF ; therefore the radius of curvature will vary inversely as the cube of PF , or directly as the cube of the normal which is terminated by either of the axes, Cor. 2. Prop. 55.

P R O P. LXXIX.

- FIG. 81, If the right line PY touches a conic section
 82, in any point P , and PS be drawn from
 83. the point of contact to the focus, and SY from the focus perpendicular to the tangent; the radius of the circle of curvature at the point P will be to half the latus rectum in the triplicate ratio of SP to SY .

- F**IRST, let the conic section be a parabola; and because a fourth part of the latus rectum is a third proportional to SP , SY , Prop. 45. and half the radius of curvature is a third proportional to SP , SY , Prop. 77. if L be the latus rectum, $2SP$ is to half L in the duplicate ratio of SP to SY , and PO is to $2SP$ as SP is to SY ; therefore PO is to half L in the triplicate ratio of SP to SY .
- FIG. 82, Secondly, if the conic section be an ellipse or an
 83. hyperbola, the latus rectum is a third proportional to the transverse and conjugate axes, and the chord of curvature, which is drawn through the focus, is a third proportional to the transverse axis, and the conjugate diameter, Prop. 78. therefore
 half PW is to AC as CD^2 to AC^2 , and
 AC is to half L as AC^2 to CB^2 ; therefore
 $\frac{1}{2}PW$ is to half L as CD^2 to CB^2 , that is, Prop. 57. in the duplicate ratio of SP to SY ; and because the triangles RPW , SPY are similar, PO is to half PW as SP to SY ; therefore PO is to half L in the triplicate ratio of SP to SY .

COR.

FIG. LXXVII.

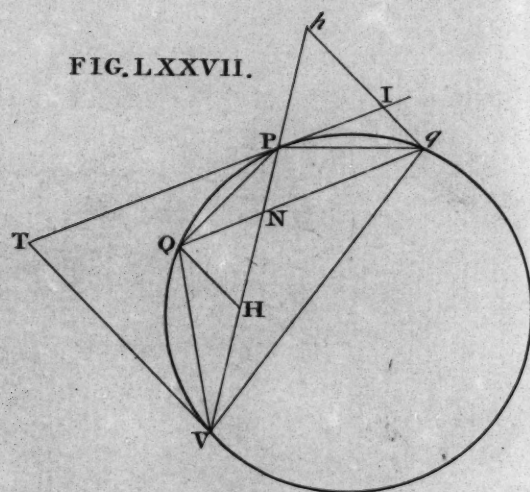


FIG. LXXVIII.

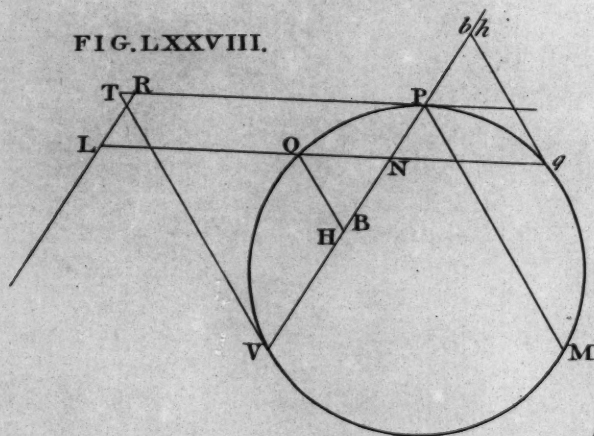


FIG. LXXIX.

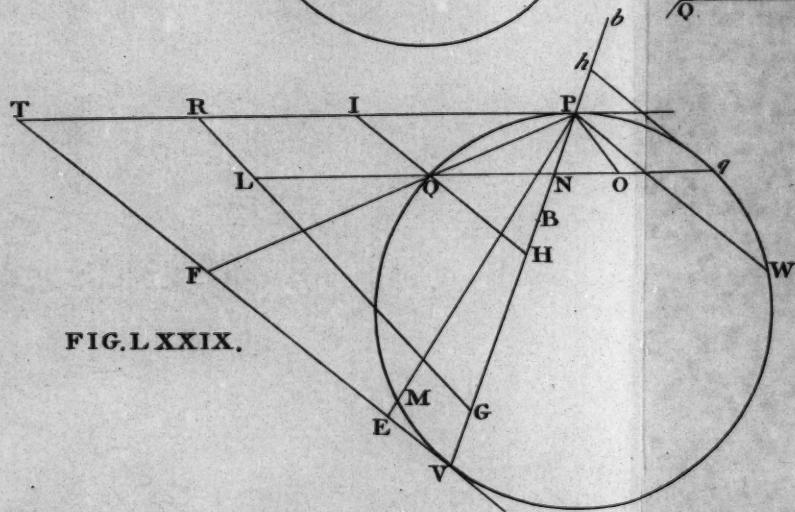


FIG. LXXX.

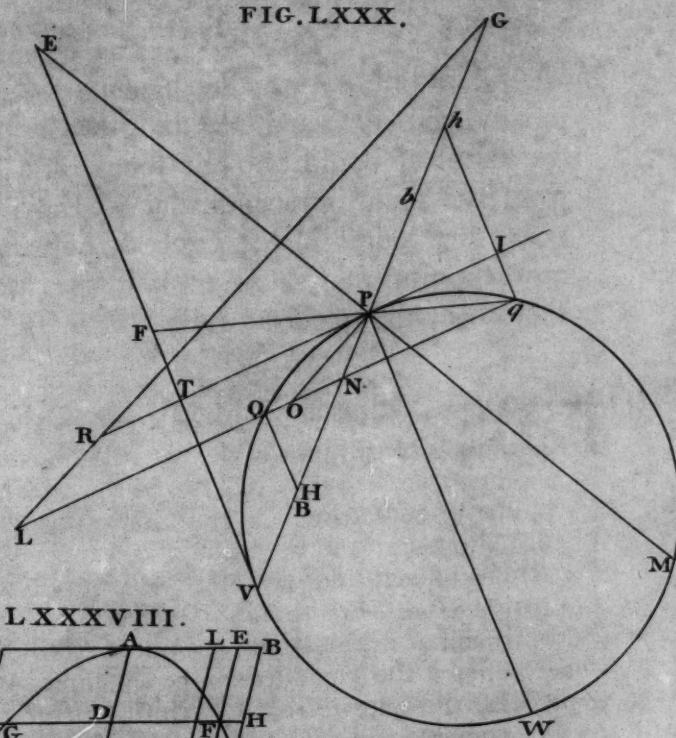


FIG. LXXXVIII.

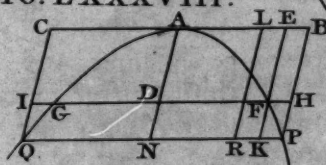
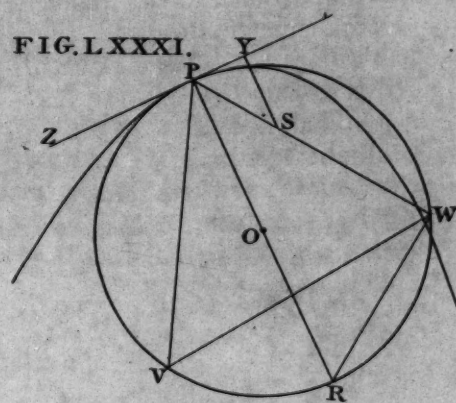


FIG. LXXXI.



COR. Hence the radius of the circle of curvature in all the conic sections is equal to $\frac{\frac{1}{2}L \times SP^3}{SY^3}$.

DEFINITIONS.

XXX. If an indefinite right line passing through any fixed point *A*, without the plane of the circle *CGB*, be carried round the whole circumference of the circle, each of the surfaces generated by this motion is called a Conical Surface. FIG. 84.

XXXI. The solid contained by the conical surface and the circle *CGB* is called a Cone.

XXXII. The point *A* is called the Vertex of the Cone.

XXXIII. The circle *CGB*, the Base of the Cone.

XXXIV. Any right line drawn from the vertex to the circumference of the base, is called a Side of the Cone.

XXXV. The right line *AD* passing through the vertex and the center of the base, which is produced indefinitely, is called the Axis.

XXXVI. A right Cone is that whose axis is perpendicular to the base.

XXXVII. A scalene Cone is that whose axis is inclined to the base.

PROP.

P R O P. LXXX.

If a cone be cut by a plane passing through its vertex, the section will be a triangle.

FIG. 84. **L**ET $ABGC$ be a cone, of which AD is the axis; and let GB be the common section of the base of the cone and the cutting plane; join AB, AG . When the generating line comes to the two points B and G , it is evident that it will coincide with the right lines AB, AG ; they are, therefore, in the surface of the cone, and they are in the plane which passes through the points A, B and G ; therefore the triangle ABG is the common section of the cone, and the plane which passes through its vertex.

P R O P. LXXXI.

If a cone be cut by a plane parallel to its base, the section will be a circle, the center of which is in the axis.

FIG. 84. **L**ET HFK be the section made by a plane parallel to the base of the cone, and let ACB, ADG be two sections of the cone, made by any two planes passing through the axis AD ; let KH, EF be the common sections of the plane HFK and the triangles ACB, ADG . Then, because the planes HFK, BGC are parallel, EH, EF will be parallel to DB, DG , and EH will be to DB as AE is to AD , or as EF is to DG ; and alternately, EH is to EF as DB to DG ; but DB is equal to DG ;

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DG ; therefore EH is equal EF , and, for the same reason, EF is equal to EK ; therefore HFK is a circle, of which E is the center.

P R O P. LXXXII.

If a scalene cone $ABDC$ be cut through the axis by a plane perpendicular to the base, making the triangle ABC , and from any point L in the right line AC , LM be drawn in the plane of the triangle, so that the angle ALM may be equal to the angle ABC , and the cone be cut by another plane passing through LM , perpendicular to the triangle ABC ; the common section $LPMQ$ of this plane and the cone will be a circle. FIG. 86.

TAKE any point N in the right line LM ; through N draw FNG parallel to CB ; and let $FPGQ$ be a section parallel to the base, passing through FG ; then the two planes $FPGQ$, $LPMQ$ being perpendicular to the plane ABC , their common section PNQ is perpendicular to FNG ; therefore PN is equal to NQ , and the square of PN equal to the rectangle FNG ; but, the angle ALM being equal to the angle ABC , or AGF , and the angles at N being vertical, the triangles FLN , MGN are similar, and MN is to NG as NF to NL ; therefore the rectangle MNL is equal to the rectangle FNG , or to the square of PN . Therefore the section $LPMQ$ is a circle, of which LM is a diameter.

This section is called a Subcontrary section.

P R O P.

P R O P. LXXXIII.

If a cone be cut by a plane, which does not pass through the vertex, and which is neither parallel to the base, nor to the plane of a subcontrary section; the common section of the plane and the surface of the cone will be an ellipse, a parabola, or an hyperbola, according as a plane passing through the vertex parallel to the cutting plane falls without the cone, touches it, or falls within the cone.

FIG. 86, 85, 87. **L**ET $ABDC$ be any cone; and let STV be the common section of a plane passing through its vertex and the plane of the base, which will fall without the base, will touch it, or it will fall within; let PMQ be a section made by a plane parallel to ASV ; through the center O of the base draw OT perpendicular to SV , meeting the circumference of the base in the points B and C ; let a plane pass through the points A, B and C , meeting the plane ASV in the line AT , the surface of the cone in AB, AC , and the plane of the section PMQ in LM ; then LM will be parallel to TA , the planes SAV, PMQ being parallel; it will meet the side AB in M , and it will meet the other side AC , Fig. 86. in L , within the cone, it will be parallel to it in Fig. 85. and it will meet it Fig. 87. produced beyond the vertex in K . Take any point N in the line LM ; let $FPGQ$ be a plane passing through N parallel to the base; and let FNG, PNQ be the common sections of this plane and the planes ABC, PMQ :

PMQ : then PNQ will be parallel to SV , and GF parallel to BT ; and BT being perpendicular to SV , FNG is perpendicular to PNQ ; therefore PN is equal to NQ , and the square of PN is equal to the rectangle FNG . First, if the line STV be without the base, through the points M and L draw MH, LK parallel to CB ; then, because the triangles LNF, LMH are similar, as also the triangles MNG, MLK , FIG. 86.

LN is to FN as LM is to HM , and

NM is to NG as LM is to LK ; therefore the rectangle LMN is to the rectangle FNG , or the square of PN , as the square of LM is to the rectangle under HM, LK ; which ratio is the same, wherever the point N be taken; therefore the section $LPMQ$ is an ellipse, of which LM is a diameter, and PNQ an ordinate, Cor. 2. Prop. 53.

Secondly, if the line STV touches the circumference of the base in C ; let DLE be the common section of the base and the plane PMQ , which is parallel to PN , and perpendicular to BLC ; and the rectangle BLC is equal to the square of DL ; therefore the square of PN is to the square of DL as the rectangle FNG to the rectangle BLC , or, because NG is equal to LC , as FN to BL ; but, the triangles MNF, MLB being similar, FN is to BL as MN to ML ; therefore the square of PN is to the square of DL as MN to ML ; and the section DME is a parabola, of which ML is a diameter, and PNQ an ordinate, Cor. 2. Prop. 44. FIG. 85.

Lastly, let the line STV fall within the base; through the vertex A draw AH parallel to GF ; and because the triangles MNF, MHA are similar, as also the triangles KNG, KHA , FIG. 87.

MN is to NF as MH is to HA , and

KN is to NG as KH is to HA ; therefore

P

the

the rectangle MNK is to the rectangle FNG ; or the square of PN , as the rectangle under MH, KH is to the square of HA , that is, in a constant ratio; therefore the section DME is an hyperbola, of which MK is a diameter, and PNQ an ordinate, Cor. 2. Prop. 30.



FIG. LXXXIII.

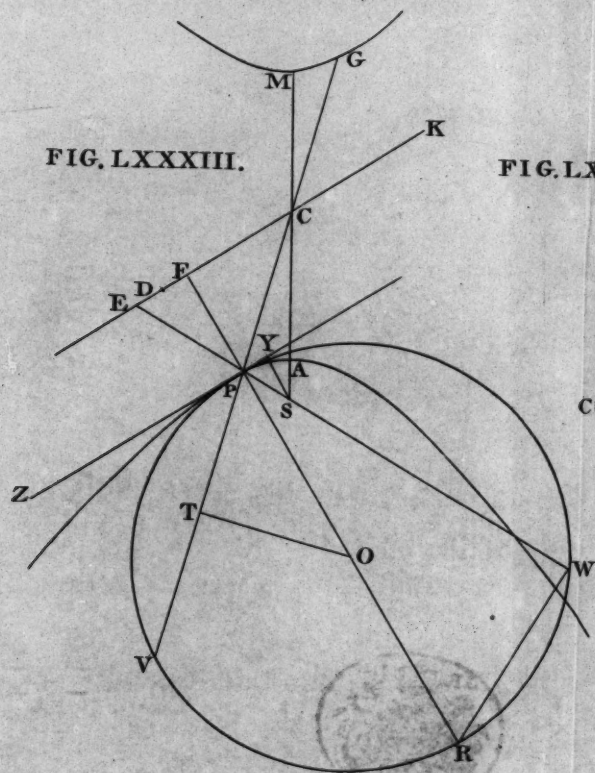


FIG. LXXXIV.

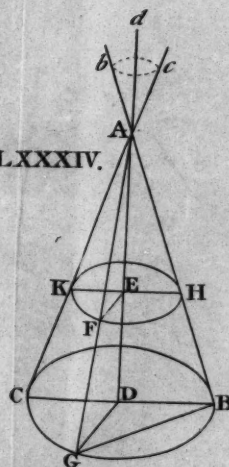


FIG. LXXXII.

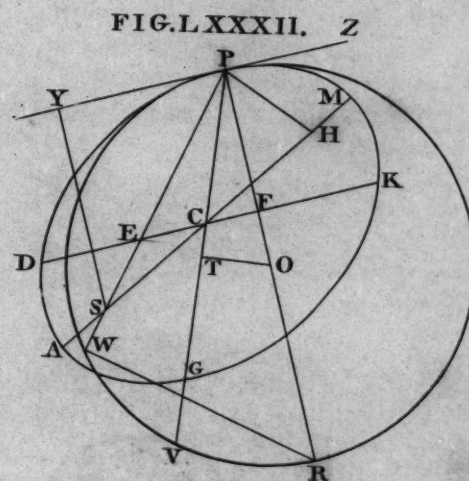


FIG. LXXXV.

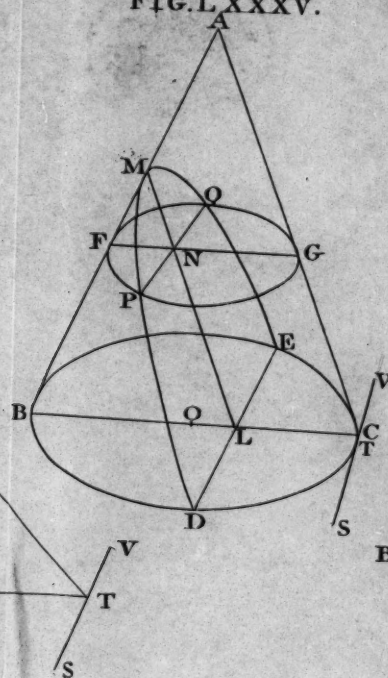


FIG. LXXXVII.

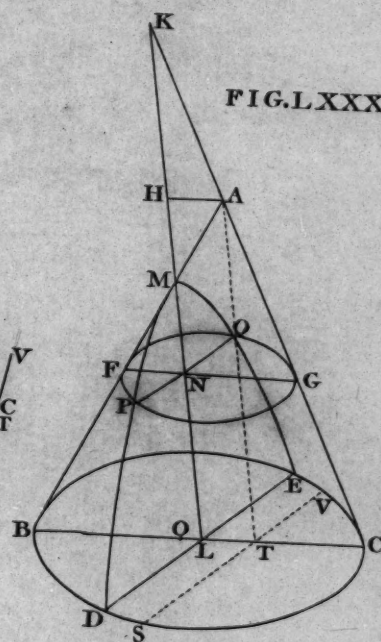
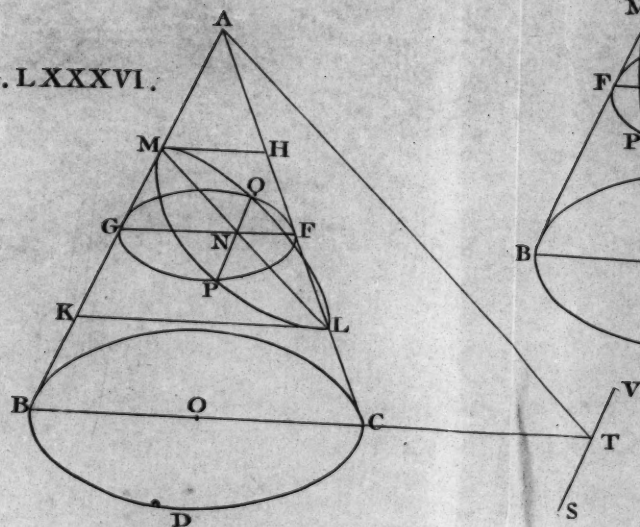


FIG. LXXXVI.



ERRATA.

PAGE 6. line 8. *for* Conjugale, *read* Conjugate.

7. line 8. *for* PROB. *read* PROB. I.

9. line 10. *for* from, *read* from.

31. line 19. *for* semi-conjugate axis, *read* conjugate semi-axis.

41. line 5. *for* parallel, *read* parallel to.

41. line 17. *for* square, *read* square of.

46. line 4. *for* equal, *read* equal to.

47. line 8. from the bottom *dele* to.

53. line 11. from the bottom, *for* VDH , *read* VDh .

89. line 6. *for* parallograms, *read* parallelograms.

93. catch word, *for* be, *read* of.

104. line 10. from the bottom, *for* tnan, *read* than.

111. line 1. *for* equal, *read* equal to.